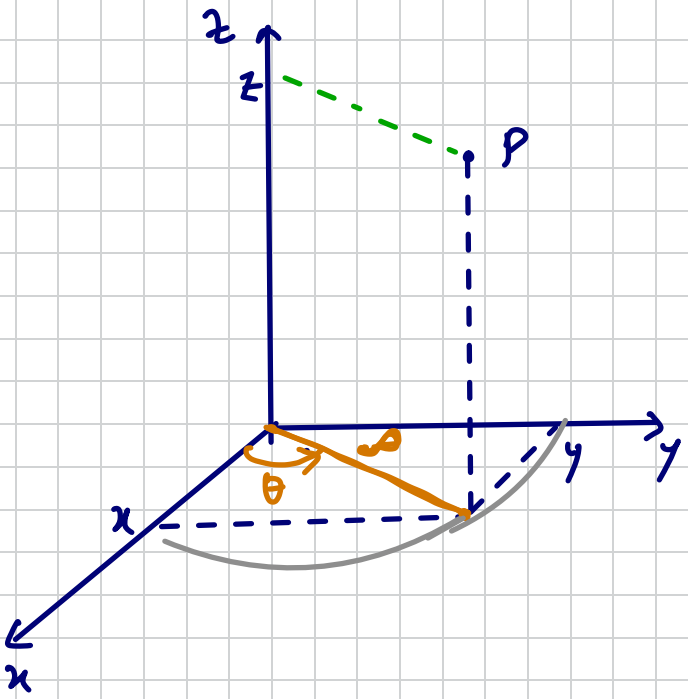
SISTEMA DE COORDENADAS CILÍNDRICAS:

É uma extensão da aplicação do sistema de coordenadas polares para o $\mathbb{R}^3$.

Um ponto $P(x, y, z)$ fica determinado pela terna ordenada $P(\rho, \theta, z)$:



ou seja, no plano xy temos o sistema polar usual.

Assim, temos:

$$\begin{cases} x = \rho \cos\theta \\ y = \rho \cdot \operatorname{sen}\theta \\ z = z \end{cases}$$

Dada $f: \Omega \subset \mathbb{R}^3 \to \mathbb{R}$,

$$\iiint_{\Omega} f(x,y,z)\, dx\,dy\,dz =$$

$$= \iiint_{\Omega'} f\big(x(\rho,\theta,z), y(\rho,\theta,z), z(\rho,\theta,z)\big) \cdot \left|\det J(T)(\rho,\theta,z)\right| d\rho\, d\theta\, dz,$$

onde

$$J(T)(\rho,\theta,z) = \begin{bmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial y}{\partial \rho} & \frac{\partial z}{\partial \rho} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial z} & \frac{\partial y}{\partial z} & \frac{\partial z}{\partial z} \end{bmatrix} = \begin{bmatrix} \cos\theta & \operatorname{sen}\theta & 0 \\ -\rho.\operatorname{sen}\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

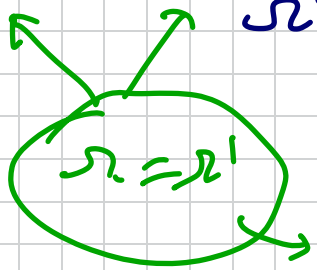
$$\Rightarrow \det J(T)(\rho,\theta,z) = \begin{vmatrix} \cos\theta & \operatorname{sen}\theta & 0 \\ -\rho\operatorname{sen}\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \rho\cos^2\theta + \rho\operatorname{sen}^2\theta = \rho\,(\underbrace{\cos^2\theta + \operatorname{sen}^2\theta}_{=1}) = \underbrace{\rho}_{>0}$$

$$\Rightarrow \left|\det J(T)(\rho,\theta,z)\right| = |\rho| = \rho$$

Dizo, a mudança para o sistema cilíndrico fica:

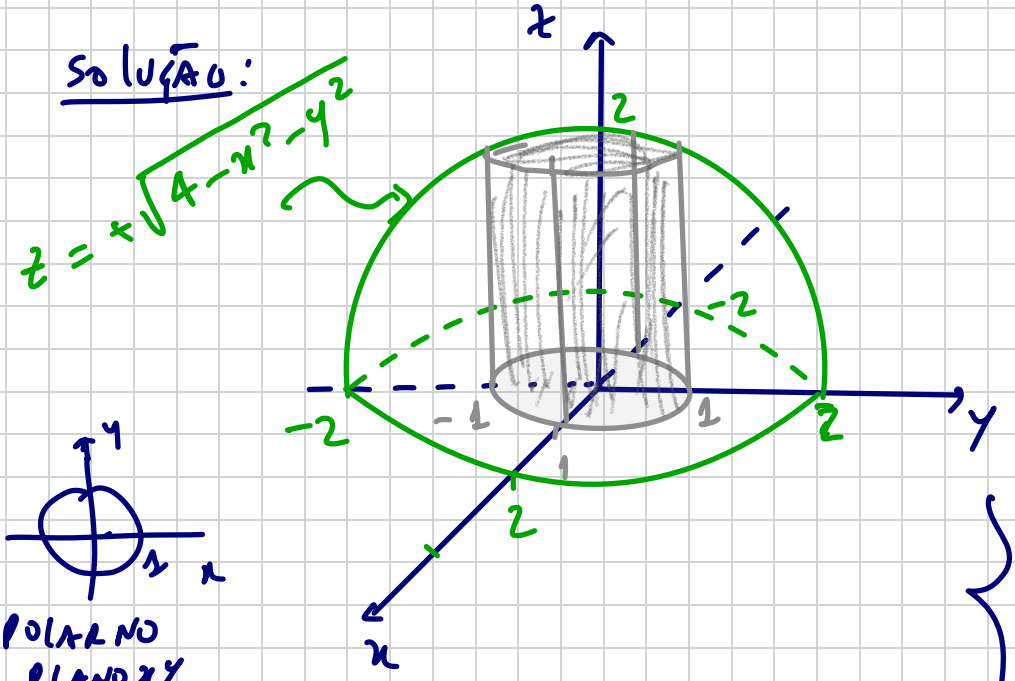
$$\iiint_{\Omega} f(x,y,z)\, dV = \iiint_{\Omega'} f(x(\rho,\theta,z), y(\rho,\theta,z), z(\rho,\theta,z)) \cdot \rho \cdot d\rho\, d\theta\, dz$$



pois a região não se deforma.

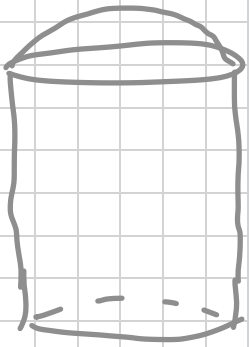
Vejamos um exemplo de aplicação:

Ex.: Calcule o volume do sólido abaixo da semi-esfera $x^2+y^2+z^2=4$, $z \geq 0$, limitado pelo cilindro $x^2+y^2=1$ e pelo plano xy, usando integrais triplas.



SOLUÇÃO:

$z = +\sqrt{4-x^2-y^2}$



$V = ?$

$$V = \iiint_{\Omega} dV = ?$$

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\operatorname{sen}\theta \\ z = z \end{cases}$$

$0 \leq \rho \leq 1$

$0 \leq \theta \leq 2\pi$

$0 \leq z \leq \sqrt{4-x^2-y^2}$

$\sqrt{4-(x^2+y^2)}$

ρ^2

$\sqrt{4-\rho^2}$

Isto posto, obtemos:

$$V = \iiint_{\Omega} dV = \int_{\theta=0}^{\theta=2\pi} \int_{\rho=0}^{\rho=1} \int_{z=0}^{z=\sqrt{4-\rho^2}} \rho \, dz\, d\rho\, d\theta =$$

DEVIDO À MUDANÇA PARA O SIST. CILÍNDRICO.

$$= \int_{\theta=0}^{\theta=2\pi} d\theta \cdot \int_{\rho=0}^{\rho=L} \rho \; z\Big|_{z=0}^{z=\sqrt{4-\rho^2}} d\rho$$

$$= \theta\Big|_0^{2\pi} \int_{\rho=0}^{\rho=L} \sqrt{4-\rho^2} \cdot \rho \, d\rho = (2\pi - 0)\left(-\frac{1}{2}\right)\int_0^1 (4-\rho^2)^{\frac{1}{2}}(-2)\rho \, d\rho$$

$$= -\frac{2\pi}{2} \cdot \frac{(4-\rho^2)^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_0^1 =$$

$$\int r^k dr$$

$$r = 4-\rho^2$$

$$dr = -2\rho \, d\rho$$

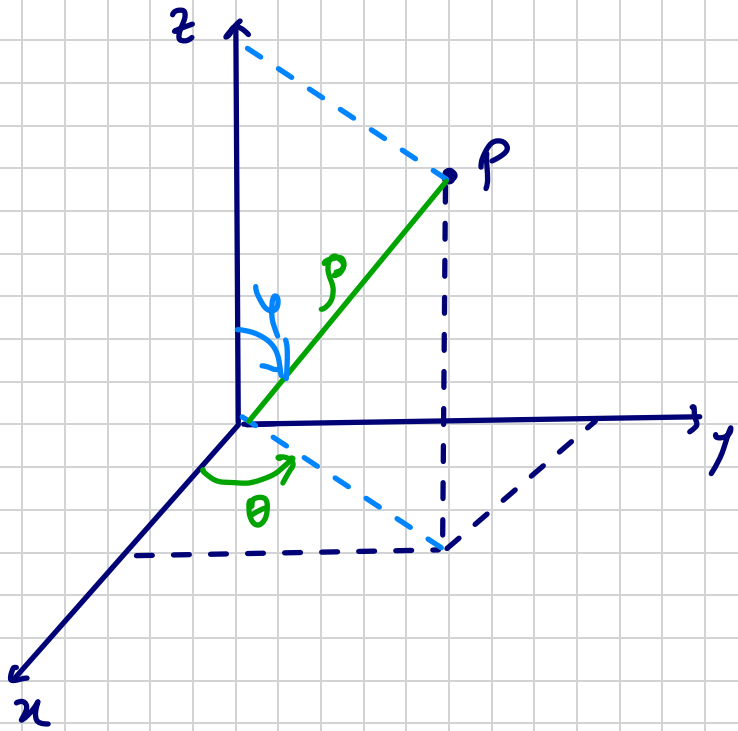
$$= -\pi \cdot \frac{2}{3}\left[(4-1)^{\frac{3}{2}} - (4-0)^{\frac{3}{2}}\right] =$$

$$= -\frac{2\pi}{3}\cdot\left(3^{\frac{3}{2}} - 4^{\frac{3}{2}}\right) = \frac{2\pi}{3}\cdot\left(4^{\frac{3}{2}} - 3^{\frac{3}{2}}\right) \text{ u.v.}$$

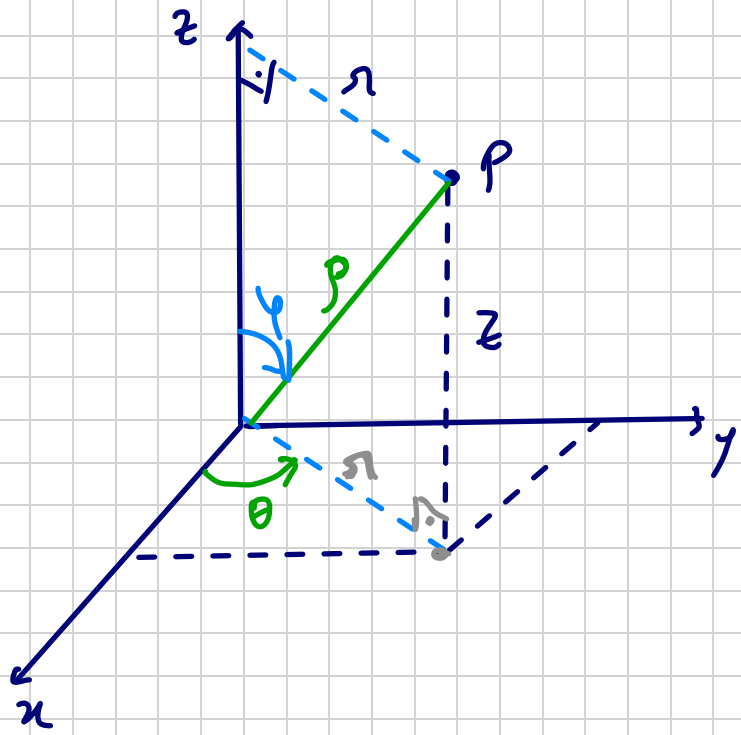
SISTEMA DE COORDENADAS ESFÉRICAS

Quando a região $\Omega \subset \mathbb{R}^3$ for esférica (ou parte de uma esfera) o melhor sistema para resolver é o de coordenadas esféricas que vamos descrever.

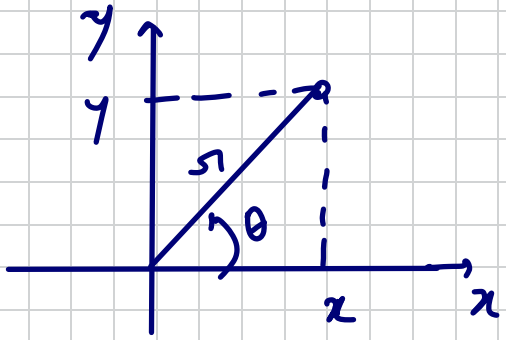
Seja $P(x,y,z)$ no $\mathbb{R}^3$.



O ponto $P(x,y,z)$ fica identificado, no sistema esférico, por $P(\rho, \varphi, \theta)$, onde ρ é a distância de P até a origem (raio da esfera); θ é o ângulo de giro; com $0 \leq \theta \leq 2\pi$ e φ é o ângulo a partir do eixo z, até o raio ρ, e é tal que $0 \leq \varphi \leq \pi$.



Seja r a projeção do raio ρ sobre o plano xy. Assim, temos, pelo sist. polar:



$$\begin{cases} x = r \cdot \cos\theta \\ y = r \cdot \operatorname{sen}\theta . \end{cases}$$

$$\operatorname{sen}\varphi = \frac{r}{\rho}$$

$$\Downarrow$$

$$r = \rho \operatorname{sen}\varphi$$

$$\begin{cases} x = \rho \operatorname{sen}\varphi \cdot \cos\theta \\ y = \rho \operatorname{sen}\varphi \cdot \operatorname{sen}\theta \end{cases}$$

$$\cos\varphi = \frac{z}{\rho} \Rightarrow z = \rho\cos\varphi$$

Portanto, dado $P(x, y, z)$ no sistema retangular, no sistema esférico, teremos $P(\rho, \varphi, \theta)$, onde:

$$\begin{cases} x = \rho \operatorname{sen}\varphi \cdot \cos\theta \\ y = \rho \operatorname{sen}\varphi \operatorname{sen}\theta \\ z = \rho \cos\varphi \end{cases}$$

Além disso, temos que:

$$x^2+y^2+z^2 = \rho^2 \operatorname{sen}^2\varphi \cos^2\theta + \rho^2 \operatorname{sen}^2\varphi \operatorname{sen}^2\theta + \rho^2 \cos^2\varphi$$

$$= \rho^2 \operatorname{sen}^2\varphi \underbrace{(\cos^2\theta + \operatorname{sen}^2\theta)}_{=1} + \rho^2 \cos^2\varphi$$

$$= \rho^2 \operatorname{sen}^2\varphi + \rho^2 \cos^2\varphi = \rho^2 \underbrace{(\operatorname{sen}^2\varphi + \cos^2\varphi)}_{=1}$$

$$= \rho^2$$

$$\Rightarrow x^2+y^2+z^2 = \rho^2$$ (uma esfera centrada na origem e raio $\rho > 0$). O que justifica o nome: coordenadas esféricas.

Vamos encontrar/adaptar a fórmula para mudar para as coordenadas esféricas.

Seja $f : \Omega \subset \mathbb{R}^3 \to \mathbb{R}$ integrável. Então, sendo

$x = \rho \operatorname{sen}\varphi \cos\theta$; $y = \rho \operatorname{sen}\varphi \operatorname{sen}\theta$ e $z = \rho \cos\varphi$;

temos:

$$j(T)(\rho,\varphi,\theta) = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial y}{\partial \rho} & \frac{\partial z}{\partial \rho} \\ \frac{\partial x}{\partial \varphi} & \frac{\partial y}{\partial \varphi} & \frac{\partial z}{\partial \varphi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \end{vmatrix} =$$

$$= \begin{bmatrix} \operatorname{sen}\varphi\cos\theta & \operatorname{sen}\varphi\operatorname{sen}\theta & \cos\varphi \\ \rho\cos\varphi\cos\theta & \rho\cos\varphi\operatorname{sen}\theta & -\rho\operatorname{sen}\varphi \\ -\rho\operatorname{sen}\varphi\operatorname{sen}\theta & \rho\operatorname{sen}\varphi\cos\theta & 0 \end{bmatrix};$$

e então:

$$\det j(T)(\rho,\varphi,\theta) =$$

$$= \begin{vmatrix} \operatorname{sen}\varphi\cos\theta & \operatorname{sen}\varphi\operatorname{sen}\theta & \cos\varphi & \operatorname{sen}\varphi\cos\theta & \operatorname{sen}\varphi\operatorname{sen}\theta \\ \rho\cos\varphi\cos\theta & \rho\cos\varphi\operatorname{sen}\theta & -\rho\operatorname{sen}\varphi & \rho\cos\varphi\cos\theta & \rho\cos\varphi\operatorname{sen}\theta \\ -\rho\operatorname{sen}\varphi\operatorname{sen}\theta & \rho\operatorname{sen}\varphi\cos\theta & 0 & -\rho\operatorname{sen}\varphi\operatorname{sen}\theta & \rho\operatorname{sen}\varphi\cos\theta \end{vmatrix}$$

$$= 0 + \rho^2\operatorname{sen}^3\varphi\operatorname{sen}^2\theta + \rho^2\cos^2\varphi\operatorname{sen}\varphi\cos^2\theta +$$

$$+ \rho^2\operatorname{sen}\varphi\cos^2\varphi\operatorname{sen}^2\theta + \rho^2\operatorname{sen}^3\varphi\cos^2\theta - 0 =$$

$$= \rho^2\operatorname{sen}^3\varphi\underbrace{(\operatorname{sen}^2\theta+\cos^2\theta)}_{1} + \rho^2\operatorname{sen}\varphi\cos^2\varphi\underbrace{(\cos^2\theta+\operatorname{sen}^2\theta)}_{1} =$$

$$= \rho^2 \operatorname{sen}^3\varphi + \rho^2 \operatorname{sen}\varphi \cos^2\varphi = \rho^2 \operatorname{sen}\varphi \underbrace{(\operatorname{sen}^2\varphi + \cos^2\varphi)}_{=1} =$$

$$= \rho^2 \operatorname{sen}\varphi$$

Logo:

$$|\det j(T)(\rho, \varphi, \theta)| = |\rho^2 \operatorname{sen}\varphi| = \rho^2 \cdot |\operatorname{sen}\varphi|$$

$$= \rho^2 \operatorname{sen}\varphi$$

$0 < \varphi \leq \pi$, e então, $\operatorname{sen}\varphi \geq 0$

Portanto, vamos obter a fórmula:

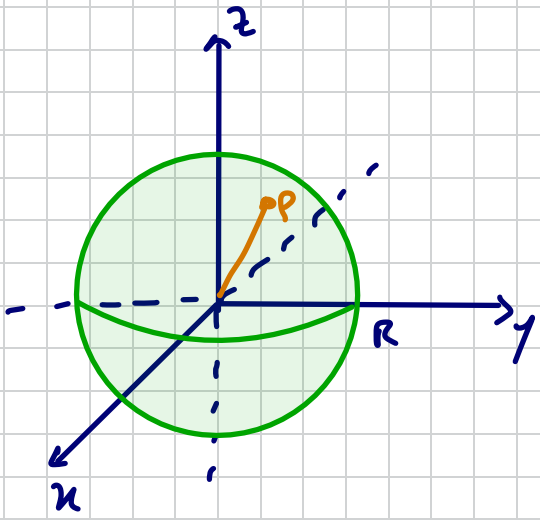
$$\iiint_{\Omega} f(x, y, z)\, dx\,dy\,dz =$$

$\Omega = \Omega'$

$$= \iiint_{\Omega'} f(x(\rho, \varphi, \theta), y(\rho, \varphi, \theta), z(\rho, \varphi, \theta)) \cdot \rho^2 \operatorname{sen}\varphi \; d\rho\, d\varphi\, d\theta.$$

Ex.: Use coordenadas esféricas para mostrar que a fórmula do volume V de uma esfera de raio R é dada por $V = \frac{4\pi R^3}{3}$.

Solução:



$P(\rho, \varphi, \theta)$

$0 \le \theta \le 2\pi$

$0 \le \rho \le R$

$0 \le \varphi \le \pi$

[ou $0 \le \varphi \le \frac{\pi}{2}$ e tomar $2 \cdot \int dV$, pela simetria]

$$V = \iiint_{\Omega} dV = \int_{\theta=0}^{\theta=2\pi} \int_{\rho=0}^{\rho=R} \int_{\varphi=0}^{\varphi=\pi} \rho^2 \operatorname{sen}\varphi \, d\varphi \, d\rho \, d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} d\theta \int_{\rho=0}^{\rho=R} \rho^2 . d\rho . \int_{\varphi=0}^{\varphi=\pi} \operatorname{sen}\varphi \, d\varphi =$$

$$= \theta \Big|_0^{2\pi} \cdot \frac{\rho^3}{3}\Big|_0^R \cdot (-\cos\varphi)\Big|_0^{\pi} =$$

$$= 2\pi \cdot \frac{R^3}{3} \cdot (\underbrace{-\cos\pi}_{1} + \underbrace{\cos 0}_{1}) = \frac{4\pi R^3}{3} //$$