06/11/26 – AULA 31

Seguiremos com outros exemplos de integração dupla.

03) Obtenha o volume do tetraedro limitado pelos planos $x+2y+z=2$, $x=2y$, $x=0$ e $z=0$ (resp.: $V=\frac{1}{3}$)

solução:

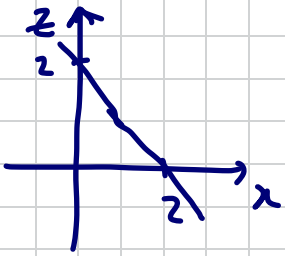
plano

$x+2y+z=2$. traços:

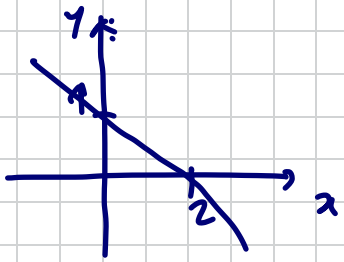
- $x=0$: $2y+z=2$



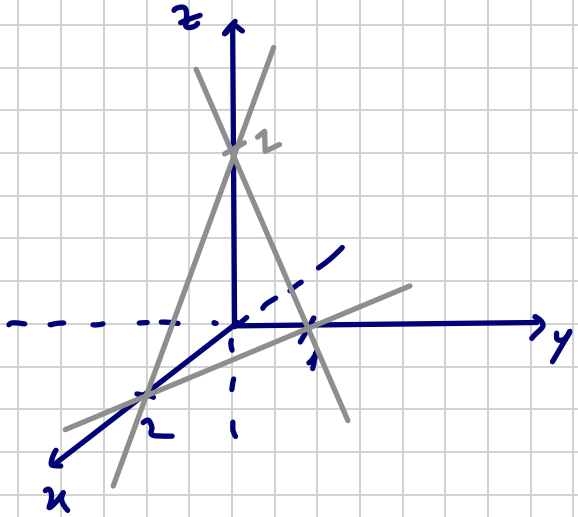
- $y=0$: $x+z=2$



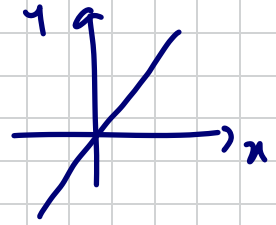
- $z=0$: $x+2y=2$

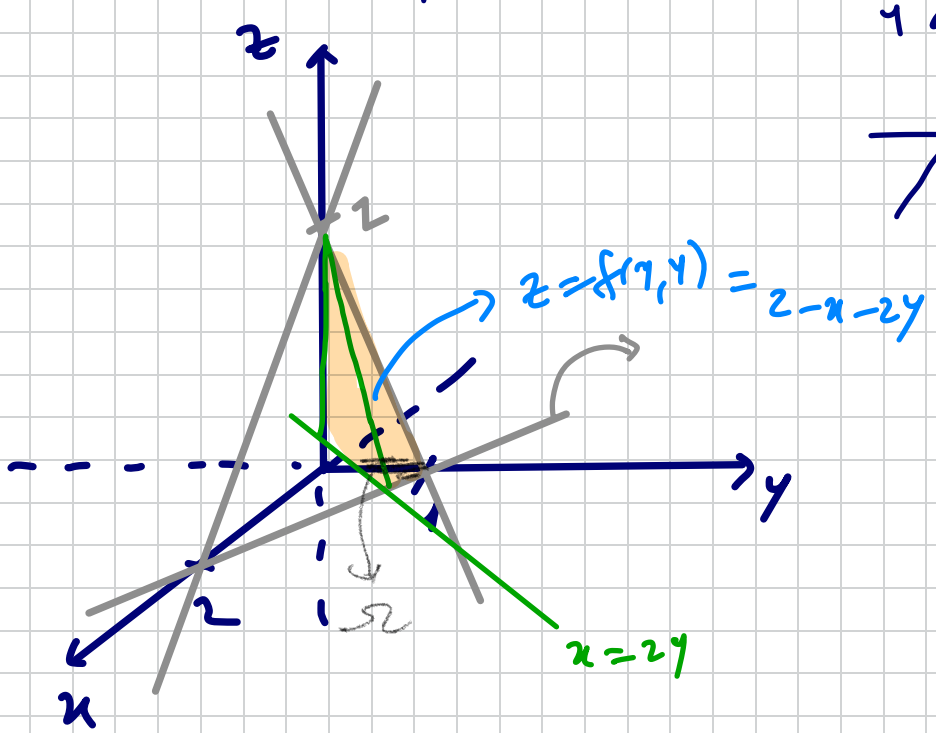


Em $\mathbb{R}^3$, juntando os 3 traços acima com os eixos coordenados, obtemos o esboço do plano $x+2y+z=2$:

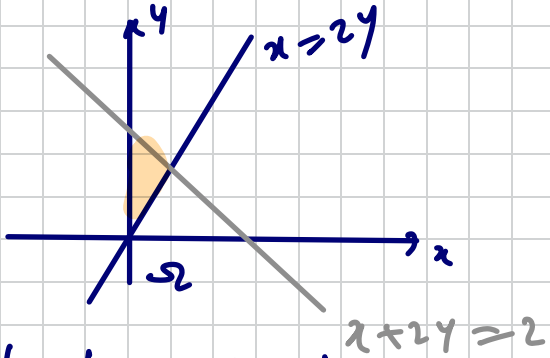


Temos ainda que obtemos o plano $x = 2y$



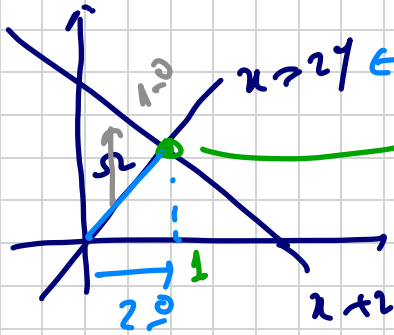


A região de integração $\Omega \subset \mathbb{R}^2$ será:



Assim, o volume V do tetraedro desejado será dado por:

$$V = \iint_{\Omega} f(x,y)\,dA = \iint_{\Omega} (2 - x - 2y)\,dA \Leftrightarrow$$



PONTO DE INTERSEÇÃO:

$x + 2y = 2$ e $x = 2y$

$x + x = 2$

$x = 1$

$2y = 2 - x$

$y = 1 - \frac{1}{2}x$

$$\int_{x=0}^{x=1}\left(\int_{y=\frac{1}{2}x}^{y=1-\frac{1}{2}x}(2-x-2y)\,dy\right)dx =$$

$$= \int_{x=0}^{x=1}\left(2y-xy-y^2\right)\Big|_{y=\frac{1}{2}x}^{y=\frac{2-x}{2}}dx =$$

$$= \int_{x=0}^{x=1}\left(\cancel{2}\left(\frac{2-x}{\cancel{2}}\right) - x\cdot\left(\frac{2-x}{2}\right) - \left(\frac{2-x}{2}\right)^2 - \right.$$

$$\left. - \left[\cancel{2}\cdot\frac{x}{\cancel{2}} - x\cdot\frac{x}{2} - \left(\frac{x}{2}\right)^2\right]\right)dx$$

$$= \int_0^1\left(2-x-x+\underbrace{\frac{x^2}{2}} - \left(\frac{4-4x+x^2}{4}\right) - x + \underbrace{\frac{x^2}{2}} + \frac{x^2}{4}\right)dx$$

$$= \int_0^1 2-2x+x^2-1+\cancel{x}-\cancel{\frac{x^2}{4}}-\cancel{x}+\cancel{\frac{x^2}{4}}\,dx$$

$$= \int_0^1\left(1-2x+x^2\right)dx = \left(x-x^2+\frac{x^3}{3}\right)\Big|_0^1 =$$

$$= \cancel{1}-\cancel{1}+\frac{1}{3}-0 = \frac{1}{3} \quad \text{u.n}$$

04) Calcule $\int_0^1 \int_x^1 \operatorname{sen} y^2 \, dy \, dx$.

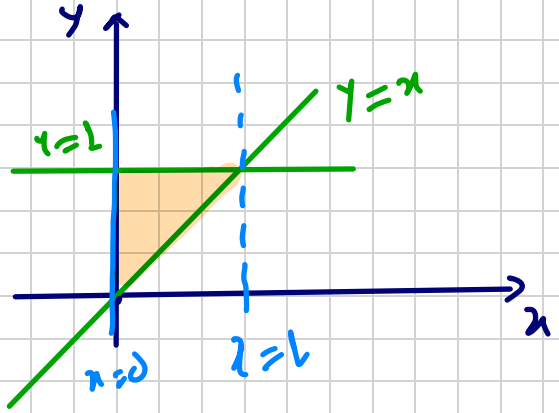
Solução:

$$\int_{x=0}^{x=1} \int_{y=x}^{y=1} \operatorname{sen} y^2 \, dy \, dx$$

↳ porém, na ordem em que está, o problema não pode ser resolvido, pois é impossível calcular $\int \operatorname{sen} y^2 \, dy$

SAÍDA: MUDAR A ORDEM DE INTEGRAÇÃO.

Vamos mudar a ordem de integração. Para isto, os limites de integração também irão mudar, para manter a mesma região de integração.



Disso:

$$\int_{x=0}^{x=1} \int_{y=x}^{y=1} \operatorname{sen} y^2 \, dy \, dx = \int_{y=0}^{y=1} \int_{x=0}^{x=y} \operatorname{sen} y^2 \, dx \, dy =$$

$$= \int_{y=0}^{y=1} \operatorname{sen} y^2 \left(\int_{x=0}^{x=y} dx \right) dy = \int_{y=0}^{y=1} \operatorname{sen} y^2 \cdot x \Big|_{x=0}^{x=y} dy =$$

$$= \frac{1}{2} \int_{y=0}^{y=1} \operatorname{sen} y^2 \, 2y \, dy = \frac{1}{2} \cdot (-\cos y^2) \Big|_{y=0}^{y=1} =$$

$\int \operatorname{sen} u \, du$

$u = y^2 \Rightarrow du = 2y\,dy$

$$= \frac{1}{2} \cdot \left(-\cos(1)^2 - (-\cos 0^2) \right) = \frac{1}{2} (-\cos 1 + 1)$$

$$= \frac{1}{2} \cdot (1 - \cos 1) \int_0^1 \int_{3y}^{3} e^{x^2} \, dx \, dy$$

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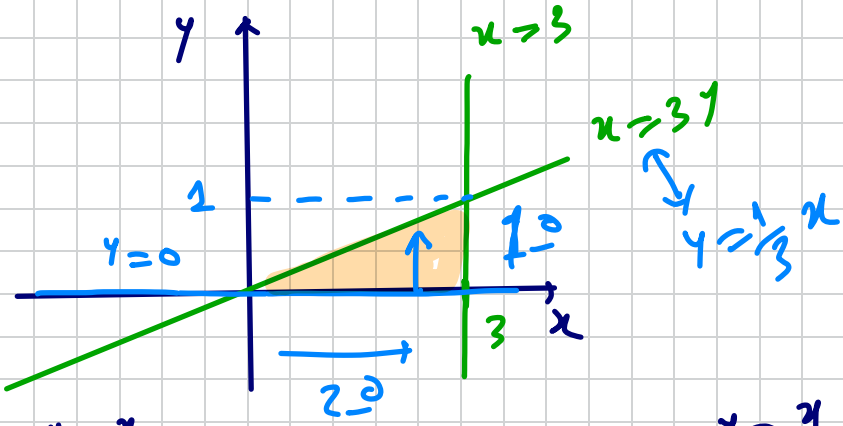
05) Calcule $\int_0^1 \int_{3y}^{3} e^{x^2} \, dx \, dy$

Solução: $\int e^{x^2} dx$ não tem como resolver.

Isto sinaliza que precisamos trocar a ordem de integração, preservando a região de integração.

Assim, teremos:

$$\int_{y=0}^{y=1}\int_{x=3y}^{x=3} e^{x^2}\,dx\,dy = \int_{x=0}^{x=3}\int_{y=0}^{y=\frac{1}{3}x} e^{x^2}\,dy\,dx =$$



$$= \int_{x=0}^{x=3} e^{x^2}\left(\int_{y=0}^{y=\frac{x}{3}} dy\right)dx = \int_{x=0}^{x=3} e^{x^2}\cdot y\Big|_{y=0}^{y=\frac{x}{3}}\,dx =$$

$$= \int_{x=0}^{x=3} e^{x^2}\cdot\frac{x}{3}\,dx = \frac{1}{3}\cdot\frac{1}{2}\int_{0}^{3} e^{x^2}\,2x\,dx =$$

$\int e^u du$

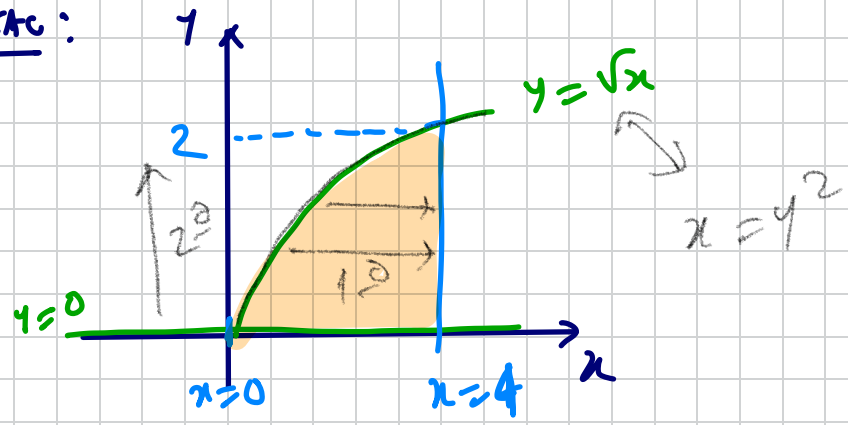
$u = x^2 \rightarrow du = 2x\,dx$

$$= \frac{1}{6}\cdot e^{x^2}\Big|_{x=0}^{x=3} = \frac{1}{6}\left(e^{(3)^2} - e^0\right) = \frac{e^9 - 1}{6}$$

06) Esboce a região de integração e faça a mudança de ordem de integração para

$$\int_0^4 \int_0^{\sqrt{x}} f(x,y)\,dy\,dx.$$

SOLUÇÃO:



$$\int_{x=0}^{x=4} \int_{y=0}^{y=\sqrt{x}} f(x,y)\,dy\,dx = \int_{y=0}^{y=2} \int_{x=y^2}^{x=4} f(x,y)\,dx\,dy$$

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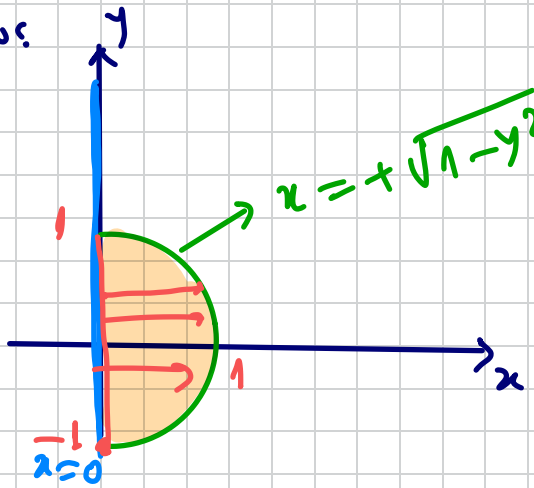
07) Calcule $\iint_{\Omega} xy^2\,dA$, onde Ω é a região limitada por $x=0$ e $x=\pm\sqrt{1-y^2}$.

SOLUÇÃO:



$x^2 = 1 - y^2$

$x^2 + y^2 = 1$

Disso, teremos:

$$\iint_{\Omega} x y^2 \, dA = \int_{y=-1}^{y=1} \int_{x=0}^{x=\sqrt{1-y^2}} x y^2 \, dx \, dy =$$

$$= \int_{y=-1}^{y=1} y^2 \left(\int_{x=0}^{x=\sqrt{1-y^2}} x \, dx \right) dy = \int_{y=-1}^{y=1} y^2 \cdot \left. \frac{x^2}{2} \right|_{0}^{\sqrt{1-y^2}} dy =$$

$$= \int_{y=-1}^{y=1} y^2 \cdot \frac{1-y^2}{2} \, dy = \frac{1}{2} \int_{y=-1}^{y=1} (y^2 - y^4) \, dy$$

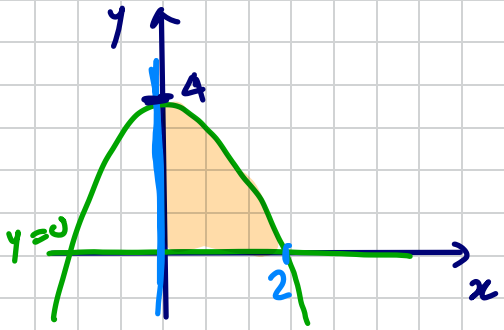
$$= \frac{1}{2} \left. \left(\frac{y^3}{3} - \frac{y^5}{5} \right) \right|_{-1}^{1} = \frac{1}{2} \left[\left(\frac{1}{3} - \frac{1}{5} - \left(\frac{-1}{3} + \frac{1}{5} \right) \right] =$$

$$= \frac{1}{2} \cdot \left[\frac{2}{3} - \frac{2}{5} \right] = \frac{1}{2} \cdot \frac{10-6}{15} = \frac{1}{\cancel{2}} \cdot \frac{\cancel{4}^{\,2}}{15} = \frac{2}{15}$$

08) Calcule: $\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y} dydx$

Solução:

$$\int_{x=0}^{x=2} \int_{y=0}^{y=4-x^2} \frac{x \cdot e^{2y}}{4-y} \, dy\,dx = \int_{y=0}^{y=4} \int_{x=0}^{x=\sqrt{4-y}} \frac{x \cdot e^{2y}}{4-y} \, dx\,dy \equiv$$



$$x^2 = 4-y$$
$$x = \sqrt{4-y}$$

$$\equiv \int_{y=0}^{y=4} \frac{e^{2y}}{4-y} \left(\int_{x=0}^{x=\sqrt{4-y}} x \, dx \right) dy =$$

$$= \int_{y=0}^{y=4} \frac{e^{2y}}{4-y} \left. \frac{x^2}{2} \right|_{x=0}^{x=\sqrt{4-y}} dy =$$

$$= \int_{y=0}^{y=4} \frac{e^{2y}}{4-y} \cdot \frac{1}{2} \left((\sqrt{4-y})^2 - 0^2 \right) dy =$$

$$= \int_{y=0}^{y=4} \frac{e^{2y}}{4-y} \cdot \frac{1}{2} \cdot (4-y) \, dy = \frac{1}{2} \cdot \frac{1}{2} \int_0^4 e^{2y} \cdot 2 \, dy =$$

$\int e^w dw$
$w = 2y \Rightarrow dw = 2dy$

$$= \frac{1}{4} \left. e^{2y} \right|_0^4 = \frac{1}{4} (e^8 - 1)$$