23/10/26 – AULA 17

MUDANÇA GERAL DE COORDENADAS NO $\mathbb{R}^3$:

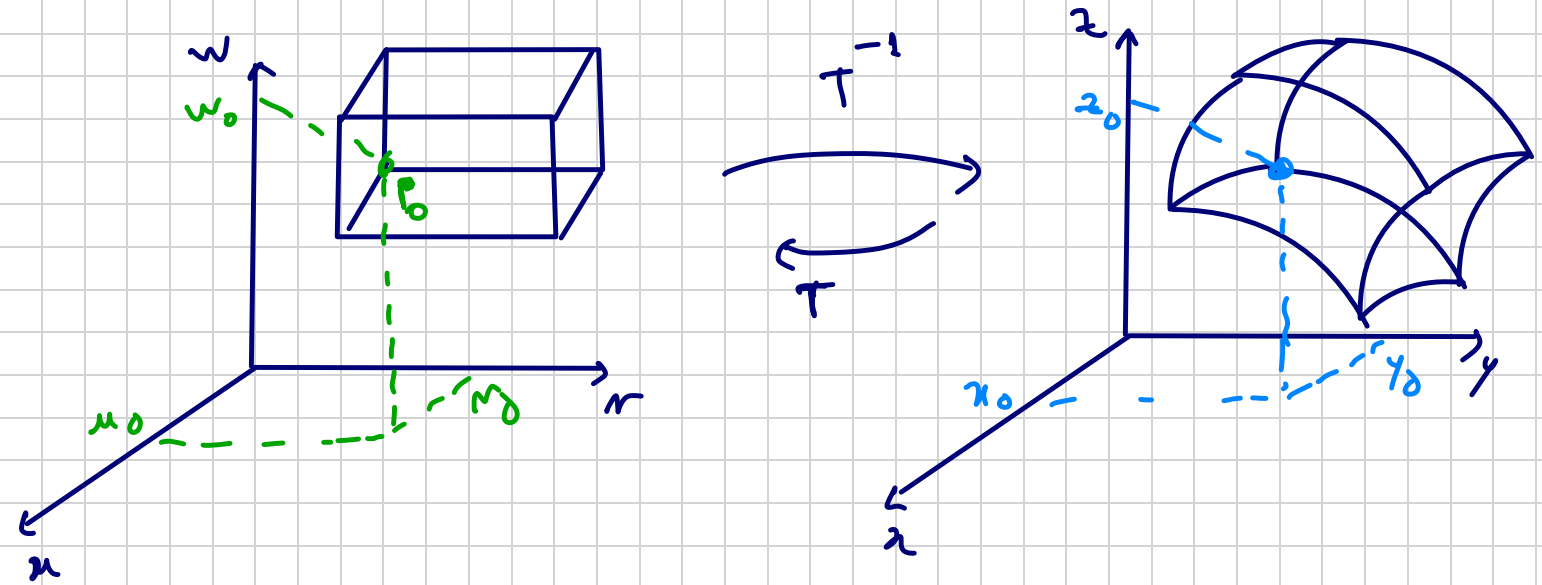
Seja $f: \Omega \subset \mathbb{R}^3 \to \mathbb{R}$ uma função integrável e considere $T: \Omega \subset \mathbb{R}^3 \to T(\Omega) \subset \mathbb{R}^3$ uma transformação injetiva e de classe C^1.

→ derivadas parciais contínuas

$$T(x, y, z) = (u, v, w), \text{ onde}$$

$$\begin{cases} x = x(u, v, w) \\ y = y(u, v, w) \\ z = z(u, v, w) \end{cases}$$

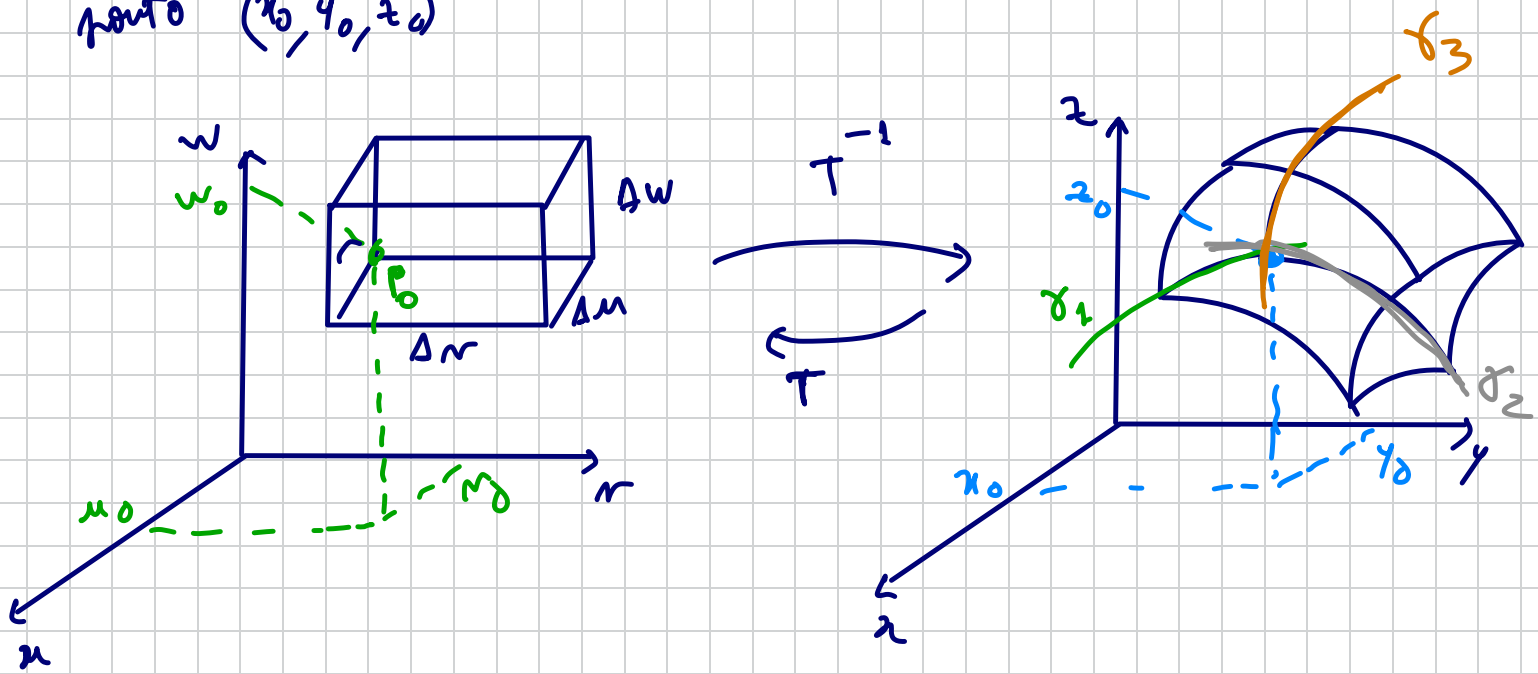
Seja A um bloco no sistema uvw, de lados $\Delta u, \Delta v$ e Δw, considere $T(A)$ o "bloco retorcido" no sistema xyz.



Seja $P_0(u_0, v_0, w_0)$ o vértice indicado na ilustração acima. Então via a transformação T, obtemos

$$T^{-1}(P_0) = (x_0, y_0, z_0)$$

Sejam γ_1, γ_2 e γ_3 as curvas nas arestas do paralelepípedo retorcido no sistema xyz, passando pelo ponto (x_0, y_0, z_0)



$$\gamma_1 : [u_0, u_0 + \Delta u] \to \mathbb{R}^3$$

$$\gamma_1(u, v_0, w_0) = (x(u, v_0, w_0), y(u, v_0, w_0), z(u, v_0, w_0))$$

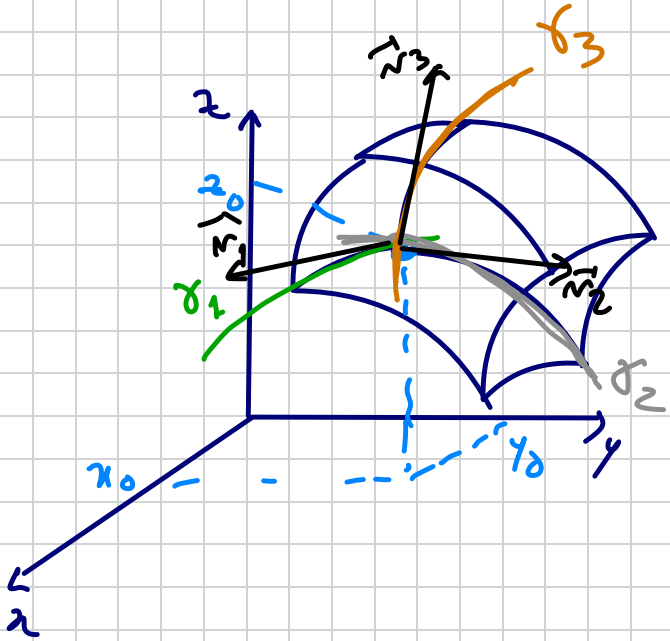
$$\gamma_2 : [v_0, v_0 + \Delta v] \to \mathbb{R}^3$$

$$\gamma_2(u_0, v, w_0) = (x(u_0, v, w_0), y(u_0, v, w_0), z(u_0, v, w_0))$$

$$\gamma_3 : [w_0, w_0 + \Delta w] \to \mathbb{R}^3$$

$$\gamma_3(u_0, v_0, w) = (x(u_0, v_0, w), y(u_0, v_0, w), z(u_0, v_0, w))$$

Sejam $\vec{v}_1$, $\vec{v}_2$ e $\vec{v}_3$ os vetores tangentes a γ_1, γ_2 e γ_3 no ponto (x_0, y_0, z_0):



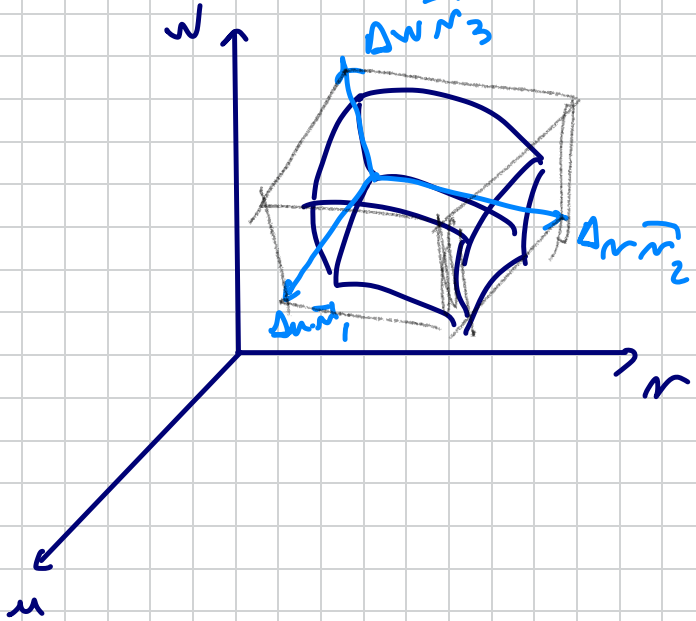
$$\vec{N}_1 = \gamma_1'(u_0) = \left(\frac{\partial x}{\partial u}(u_0, v_0, w_0), \frac{\partial y}{\partial u}(u_0, v_0, w_0), \frac{\partial z}{\partial u}(u_0, v_0, w_0) \right)$$

$$\vec{N}_2 = \gamma_2'(v_0) = \left(\frac{\partial x}{\partial v}(u_0, v_0, w_0), \frac{\partial y}{\partial v}(u_0, v_0, w_0), \frac{\partial z}{\partial v}(u_0, v_0, w_0) \right)$$

$$\vec{N}_3 = \gamma_3'(w_0) = \left(\frac{\partial x}{\partial w}(u_0, v_0, w_0), \frac{\partial y}{\partial w}(u_0, v_0, w_0), \frac{\partial z}{\partial w}(u_0, v_0, w_0) \right)$$

Redimensionando estes vetores para os comprimentos $\Delta u, \Delta v, \Delta w$, obtendo:

$\Delta u \cdot \vec{N}_1$; $\Delta v \vec{N}_2$ e $\Delta w \cdot \vec{N}_3$:



O volume $\tilde{V}$ do paralelepípedo formado pelos vetores $\Delta u \vec{v}_1$, $\Delta v \vec{v}_2$ e $\Delta w \vec{v}_3$ será dado pelo módulo do produto misto(*) entre eles, ou seja:

$$\tilde{V} = \| [\Delta u \vec{v}_1, \Delta v \vec{v}_2, \Delta w \vec{v}_3] \| \; ; \text{ onde}$$

$$[\Delta u \vec{v}_1, \Delta v \vec{v}_2, \Delta w \vec{v}_3] =$$

$$= \left[\Delta u \cdot \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right), \Delta v \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right), \Delta w \left(\frac{\partial x}{\partial w}, \frac{\partial y}{\partial w}, \frac{\partial z}{\partial w}\right) \right]$$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} \Delta u & \frac{\partial y}{\partial u} \Delta u & \frac{\partial z}{\partial u} \Delta u \\ \frac{\partial x}{\partial v} \Delta v & \frac{\partial y}{\partial v} \Delta v & \frac{\partial z}{\partial v} \Delta v \\ \frac{\partial x}{\partial w} \Delta w & \frac{\partial y}{\partial w} \Delta w & \frac{\partial z}{\partial w} \Delta w \end{vmatrix} =$$

$$= \overbrace{\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix}}^{\det(J(T)(u,v,w))} \underbrace{\Delta u}_{>0} \underbrace{\Delta v}_{>0} \underbrace{\Delta w}_{>0} =$$

$$\det \cdot (J(T)(u,v,w)) \Delta u \Delta v \cdot \Delta w$$

(*) Dados os vetores $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$ e $\vec{w} = (w_1, w_2, w_3)$, o produto misto entre $\vec{u}, \vec{v}$ e $\vec{w}$ é dado por:

$$[\vec{u}, \vec{v}, \vec{w}] = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

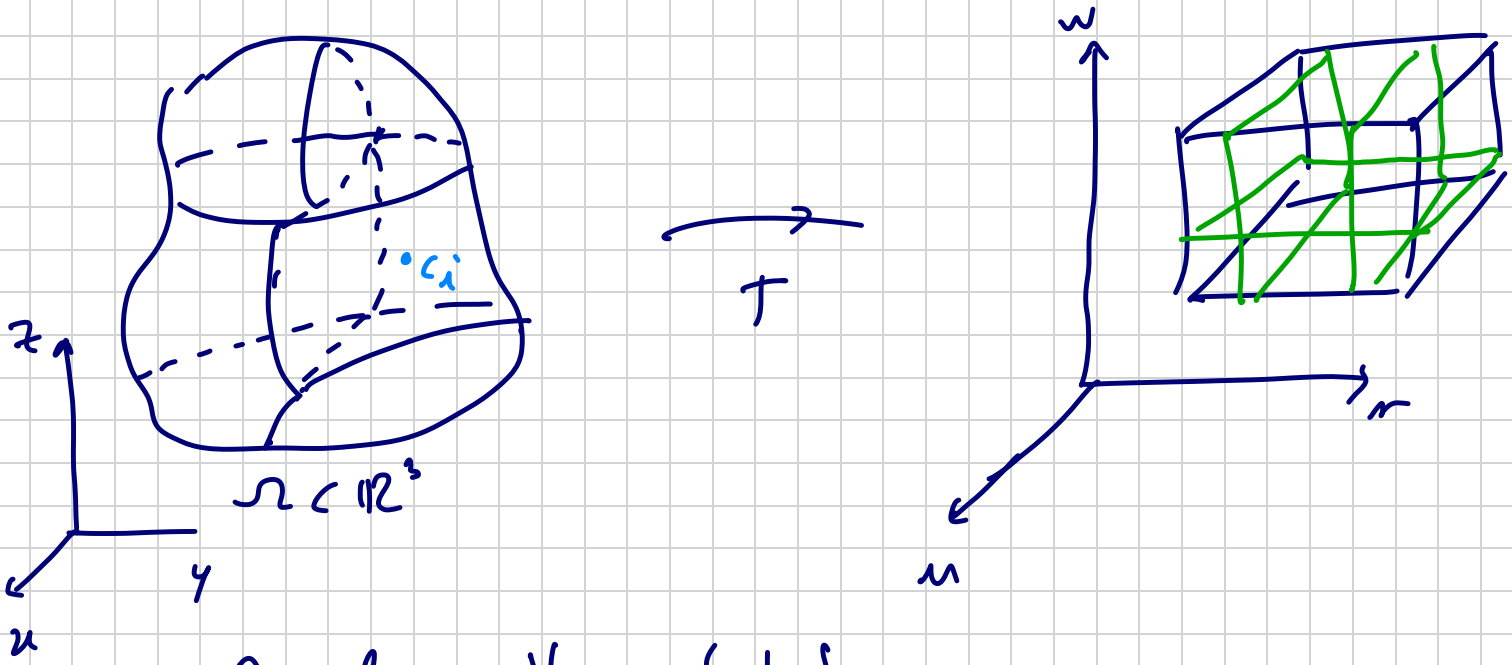
Assim, o volume $\tilde{V}$ será:

$$\tilde{V} = |\det J(T)(u,v,w)| \cdot \Delta u \cdot \Delta v \, \Delta w$$

Disso, o volume V do "paralelepípedo retorcido" é aproximadamente $\tilde{V}$, ou seja:

$$V \cong |\det J(T)(u,v,w)| \cdot \Delta u \cdot \Delta v \, \Delta w$$

Seja $f: \Omega \subset \mathbb{R}^3 \to \mathbb{R}$ integrável e considere P uma partição do Ω.



O volume V será dado por:

$$V \cong \sum_{i=1}^{\tilde{n}} f(c_i) \cdot \mathrm{Vol}(B_i) =$$

$$= \sum_{i=1}^{\tilde{n}} f(x(u_i,v_i,w_i), y(u_i,v_i,w_i), z(u_i,v_i,w_i)) \cdot |\det J(T)(u_i,v_i,w_i)| \, \Delta u_i \Delta v_i \Delta w_i$$

Tomando o limite com $n \to \infty$, vemos então:

$$\iiint_{\Omega} f(x,y,z)\,dV = \iiint_{\Omega'} f(x(u,v,w), y(u,v,w), z(u,v,w)) \cdot \left|\det J(T)(u,v,w)\right| du\,dv\,dw$$

FÓRMULA DA MUDANÇA DE VARIÁVEL NO $\mathbb{R}^3$.

Vejamos um exemplo:

Ex.: Calcule $\iiint_{\Omega} \frac{e^{x-y+z}}{x+y-z}\,dx\,dy\,dz$, sendo Ω a região

dada por: $0 \le \underbrace{x-y+z}_{u} \le 1$; $1 \le \underbrace{x+y-z}_{v} \le 2$ e

$0 \le \underbrace{z}_{w} \le 1$.

SOLUÇÃO: Escreva $\begin{cases} u = x-y+z \\ v = x+y-z \\ w = z \end{cases}$ $\longrightarrow$ $\boxed{z = w}$

$u + v = 2x$

$\Rightarrow \boxed{x = \frac{1}{2}u + \frac{1}{2}v}$

$v = x + y - z$

$\Rightarrow y = v - x + z$

$y = v - \left(\frac{1}{2}u + \frac{1}{2}v\right) + w$

$\boxed{y = -\frac{1}{2}u + \frac{1}{2}v + w}$

$$J(T)(u,v,w) = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & +\frac{1}{2} & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\Rightarrow \det J(T)(u,v,w) = \begin{vmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & +\frac{1}{2} & 0 \\ 0 & 1 & 1 \end{vmatrix} \begin{matrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & +\frac{1}{2} \\ 0 & 1 \end{matrix} =$$

$$= +\frac{1}{4} + 0 + 0 - 0 - 0 - \left(-\frac{1}{4}\right) = \frac{1}{2}.$$

Assim, obtemos:

$$\iiint_{\Omega} \frac{e^{\overbrace{x-y+z}^{u}}}{\underbrace{x+y-z}_{v}} \, dx\,dy\,dz = \iiint_{\Omega'} \frac{e^u}{v} \cdot \underbrace{|\det J(T)(u,v,w)|}_{\frac{1}{2}} \, du\,dv\,dw$$

$$= \frac{1}{2} \int_{w=0}^{w=1} \int_{v=1}^{v=2} \int_{u=0}^{u=1} \frac{e^u}{v} \, du\,dv\,dw =$$

$$= \frac{1}{2} \int_{w=0}^{w=1} dw \int_{v=1}^{v=2} \frac{dv}{v} \cdot \int_{u=0}^{u=1} e^u du$$

$$= \frac{1}{2} \cdot w\Big|_0^1 \cdot \ln|v|\Big|_1^2 \cdot e^u\Big|_0^1 =$$

$$= \frac{1}{2} \cdot (1-0) \cdot \underbrace{\left(\ln|2| - \underbrace{\ln|1|}_{=0}\right)}_{\ln 2} \cdot \underbrace{\left(e^1 - e^0\right)}_{1} =$$

$$= \frac{(e-1)\cdot \ln 2}{2} \quad =$$