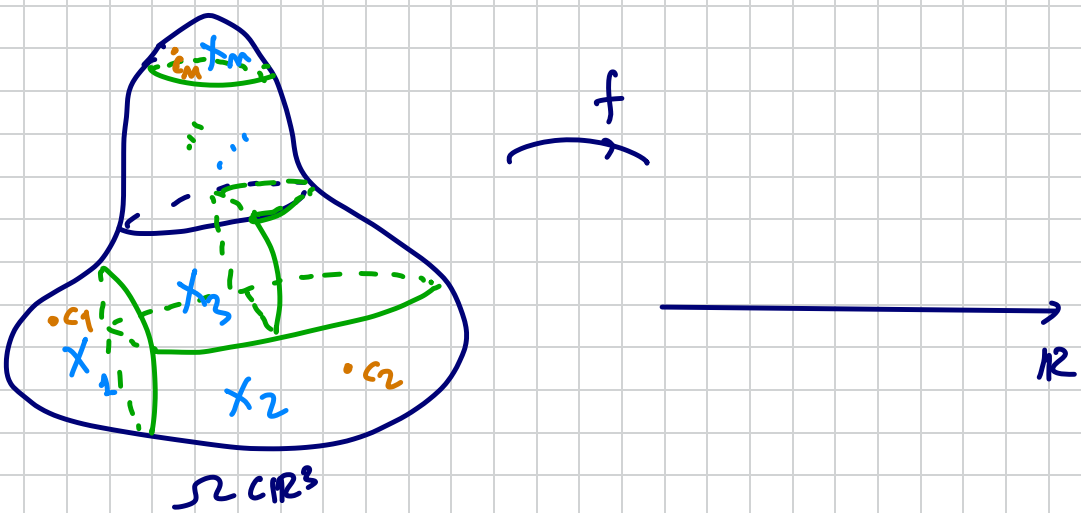
INTEGRAIS TRIPLAS:

Seja $w = f(x,y,z)$ uma função definida em um conjunto $\mathcal{J}$-mensurável $\Omega \subset \mathbb{R}^3$, $f: \Omega \subset \mathbb{R}^3 \to \mathbb{R}$, com Ω limitado.

Seja $D = \{X_1, X_2, \dots, X_n\}$ uma decomposição de Ω, ou seja:

$$\Omega = X_1 \cup X_2 \cup \dots \cup X_n \quad ; \quad e$$

$$\text{int}(X_i) \cap \text{int}(X_j) = \emptyset, \text{ se } i \neq j.$$



Tome $c_i \in X_i$, $i \in \{1, 2, \dots, n\}$

Definimos a decomposição pontilhada por

$$\Sigma(f; D) = \sum_{i=1}^{n} f(c_i) \cdot \text{Vol}(X_i)$$

Sendo $M_i = \sup_{x \in X_i} f(x)$ e $m_i = \inf_{x \in X_i} f(x)$,

então; Dada uma partição P de Ω,

$$m_i \leq f(x) \leq M_i$$

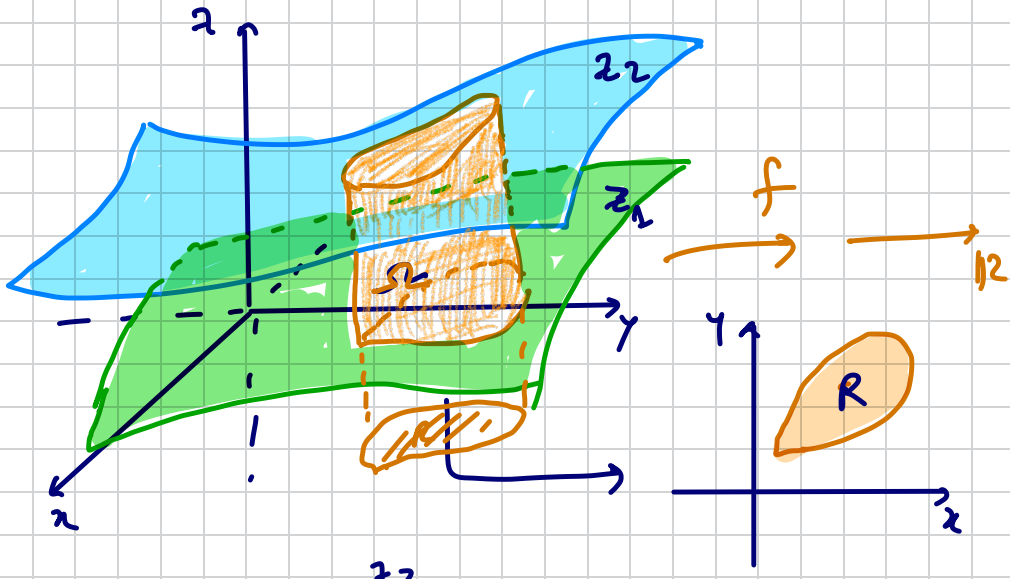
$$\Rightarrow s(f; P) \leq \Sigma(f; D) \leq S(f; P)$$

No limite, com $n \to \infty$, obtemos, analogamente a duas variáveis, o conceito do integral triplo, ou seja:

$$\iiint_{\Omega} f(x,y,z)\, dx dy dz = \lim_{n \to \infty} \Sigma(f; D) =$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} f(c_i) \cdot \text{Vol}(X_i)$$

Na prática, uma integral tripla é decomposta por uma integral dupla combinada com uma integral simples, pois $\Omega \subset \mathbb{R}^3$ geralmente fica limitada entre duas superfícies. Por exemplo:



$$\iiint_{\Omega} f = \iint_{R} dx\,dy \int_{z_1}^{z_2} f(x,y,z)\,dz$$

z_1

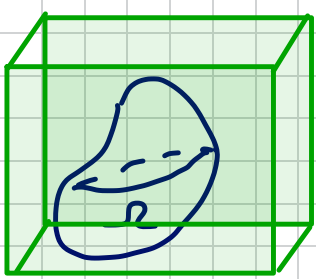
$g(x,y)$

O volume V da região Ω é dado por:

$$V = \iiint_{\Omega} dV = \iiint_{\Omega} dx\,dy\,dz$$

[OU OUTRA ORDEM CONVENIENTE]

De fato, seja $A \subset \mathbb{R}^3$ um bloco (PARALELEPÍPEDO) tal que $\Omega \subset A$.



Então:

$$V = \iiint_{A} \xi_{\Omega}(x,y,z)\,dx\,dy\,dz =$$

$$= \iiint_{\Omega} \underbrace{\xi_{\Omega}(x,y,z)}_{1}\,dx\,dy\,dz + \iiint_{A\setminus\Omega} \underbrace{\xi_{\Omega}(x,y,z)}_{=0}\,dx\,dy\,dz$$

$$= 0$$

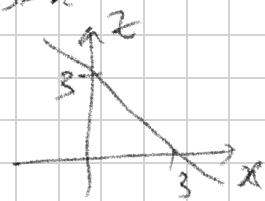
$$= \iiint_{\Omega} dx\,dy\,dz = \iiint_{\Omega} dV.$$

No que segue faremos alguns exemplos:

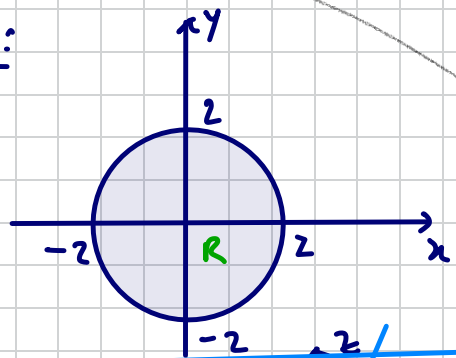
01) Ache o volume V do sólido obtido pela região cortada do cilindro $x^2+y^2=4$ pelos planos $z=0$ e $x+z=3$.

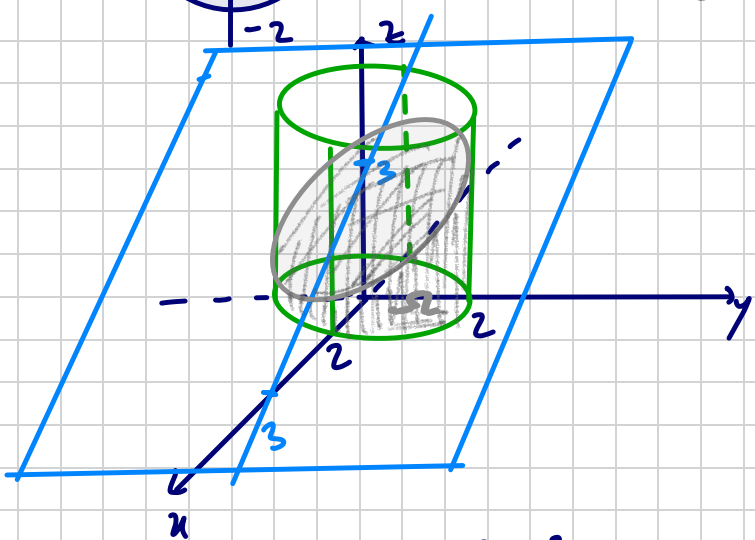
PLANO xy

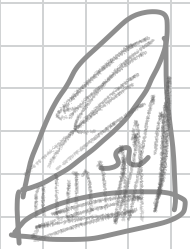
$z=3-x$



Solução:







$$V = \iiint_{\Omega} dV = \underbrace{\iint_{R} dx\,dy}_{} \int_{z=0}^{z=3-x} dz =$$



→ COORD. POLARES

$$= \iint_R dx\,dy \int_0^{3-x} z = \iint_R (3-x)\,dx\,dy =$$

$x = \rho\cos\theta \quad dx\,dy = \rho\,d\rho\,d\theta.$
$y = \rho\sin\theta.$
$0 < \rho \leq 2$
$0 \leq \theta \leq 2\pi$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{\rho=0}^{\rho=2} (3 - \rho\cos\theta)\cdot\rho\,d\rho\,d\theta =$$

$$= \int_{\theta=0}^{\theta=2\pi} \left(\int_{\rho=0}^{\rho=2} (3\rho - \rho^2\cos\theta)\,d\rho \right) d\theta = \int_{\theta=0}^{\theta=2\pi} \left(\frac{3\rho^2}{2} - \frac{\rho^3}{3}\cos\theta \right)\Big|_{\rho=0}^{\rho=2} d\theta =$$

$$= \int_{\theta=0}^{\theta=2\pi} \left(6 - \frac{8}{3}\cos\theta\right) d\theta = \left(6\theta - \frac{8}{3}\operatorname{sen}\theta\right)\Big|_{\theta=0}^{\theta=2\pi} =$$

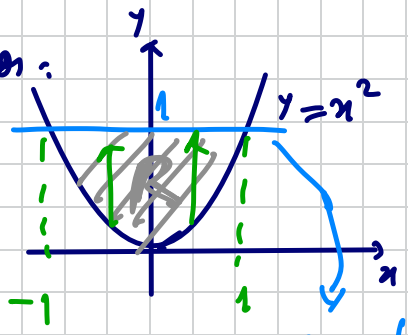
$$= 6\cdot(2\pi) - \underbrace{\frac{8}{3}\cdot\operatorname{sen} 2\pi}_{=0} - \left[\underbrace{6\cdot 0 - \frac{8}{3}\operatorname{sen} 0}_{=0}\right] = 12\pi \text{ u.v.}$$

u.v. ↓ UNIDADES DE VOLUME.

02) Use integral tripla para determinar o volume do sólido limitado pelo cilindro $y = x^2$ e pelos planos $z = 0$ e $y + z = 1$.

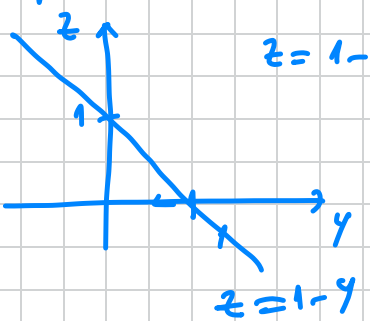
solução:

no plano xy, temos:

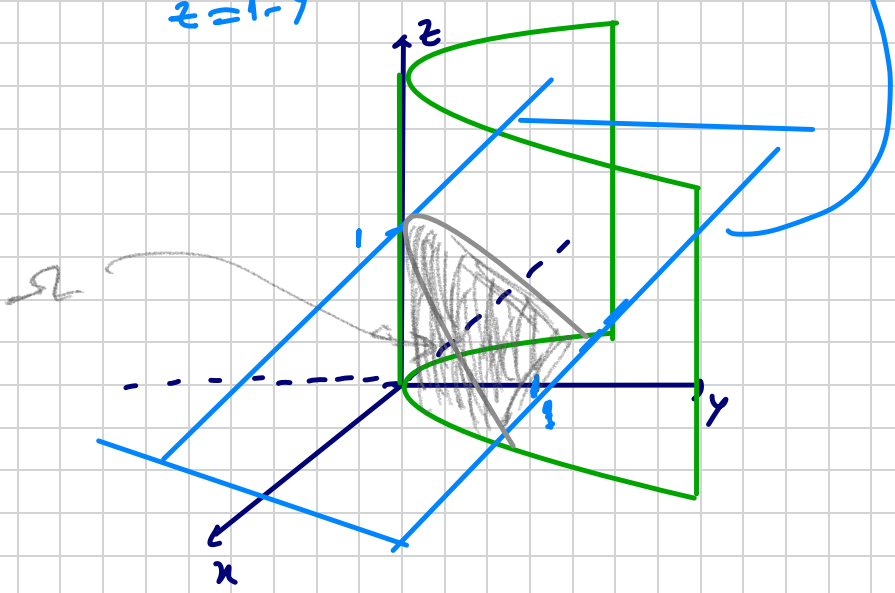


no plano yz temos a reta:

$z = 1 - y$



quando $z = 0$



Pela ilustração, obtemos:

$$V = \iiint_{\Omega} dV = \iint_{R} dx\,dy \int_{z=0}^{z=1-y} dz =$$

$$= \int_{x=-1}^{x=1} \int_{y=x^2}^{y=1} dy\,dx \int_{z=0}^{z=1-y} dz = 2 \cdot \int_{x=0}^{x=1} \int_{y=x^2}^{y=1} dy\,dx \int_{z=0}^{z=1-y} dz$$

DEVIDO À SIMETRIA

$$= 2 \cdot \int_{x=0}^{x=1} \int_{y=x^2}^{y=1} dy\,dx \cdot (z)\Big|_{0}^{1-y} = 2 \int_{x=0}^{x=1} \int_{y=x^2}^{y=1} (1-y)\,dy\,dx$$

$$= 2 \int_{x=0}^{x=1} \left(y - \frac{y^2}{2}\right)\Big|_{y=x^2}^{y=1} dx = 2 \int_{x=0}^{x=1} \left(1 - \frac{1}{2} - \left[x^2 - \frac{x^4}{2}\right]\right) dx$$

$$= 2\int_0^1 \left(\frac{1}{2} - x^2 + \frac{x^4}{2}\right)dx = \int_0^1 \left(1 - 2x^2 + x^4\right)dx$$

$$= \left(x - \frac{2x^3}{3} + \frac{x^5}{5}\right)\Bigg|_0^1 = 1 - \frac{2}{3} + \frac{1}{5} - 0$$

$$= \frac{15 - 10 + 3}{15} = \frac{8}{15} \text{ u. v.}$$

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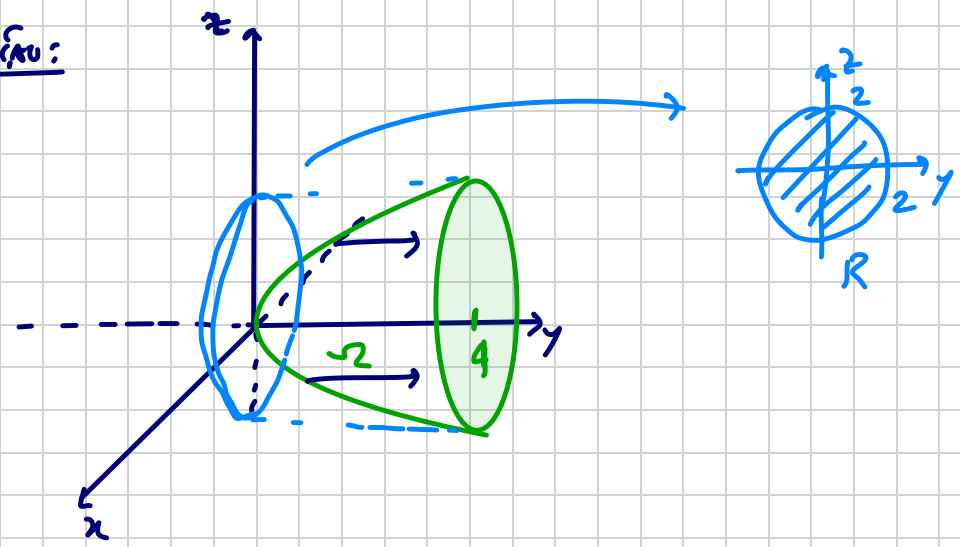
03) Calcule $\iiint_\Omega \sqrt{x^2+z^2}\, dV$, onde Ω é a região limitada pelo paraboloide $y = x^2 + z^2$ e pelo plano $y = 4$.

solução:



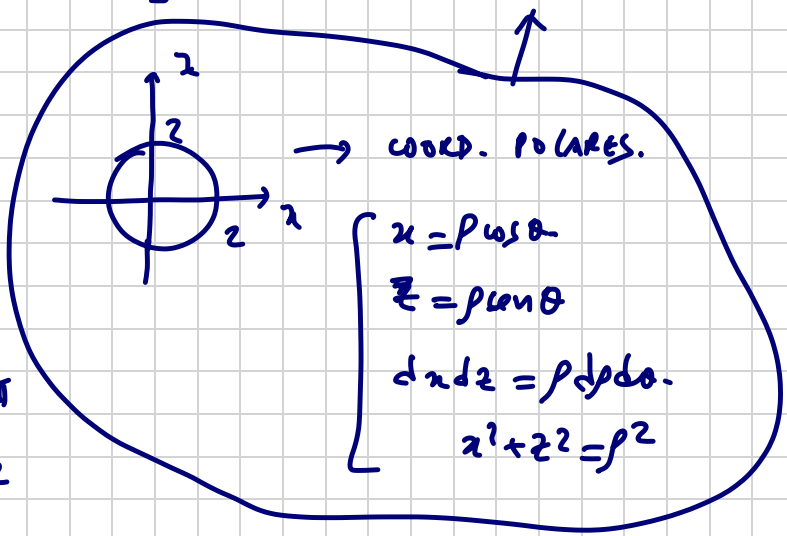
$$\iiint_{\Omega} \sqrt{x^2+z^2}\, dV = \iint_R dx\,dz \int_{y=x^2+z^2}^{y=4} \sqrt{x^2+z^2}\, dy =$$

CONSTANTE PARA y

$$= \iint_R \sqrt{x^2+z^2}\, dx\,dz \int_{y=x^2+z^2}^{y=4} dy =$$

$$= \iint_R \sqrt{x^2+z^2}\, dz\,dx \,(y)\Big|_{y=x^2+z^2}^{y=4} =$$

$$= \iint_R \left[4-(x^2+z^2)\right] \cdot \sqrt{x^2+z^2}\, dz\,dx =$$



$$= \int_{\theta=0}^{\theta=2\pi} \int_{\rho=0}^{\rho=2} [4-\rho^2]\cdot\rho \cdot \rho \, d\rho \, d\theta = \int_{\theta=0}^{\theta=2\pi} \int_{\rho=0}^{\rho=2} (4\rho^2 - \rho^4) \, d\rho \, d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} d\theta \cdot \int_{\rho=0}^{\rho=2} (4\rho^2 - \rho^4) \, d\rho = \theta\Big|_0^{2\pi} \cdot \left(\frac{4\rho^3}{3} - \frac{\rho^5}{5}\right)\Big|_{\rho=0}^{\rho=2} =$$

$$= 2\pi \cdot \left(\frac{32}{3} - \frac{32}{5} - 0\right) = \frac{2\pi \cdot (160 - 96)}{15} = \frac{128\pi}{15}$$