

AULA DE EXERCÍCIOS.LISTA 06

1. Determine o vetor gradiente de f no ponto P indicado:

(a) $f(x, y, z) = x^2 - 3yz + z^3$; $P(2, 1, 0)$.

(b) $f(x, y, z) = 2\sqrt{xyz}$; $P(3, -4, -3)$.

(b) $f(x, y, z) = 2 \cdot (xyz)^{\frac{1}{2}}$; $P(3, -4, -3)$

$$\nabla f(P) = \left(\frac{\partial f}{\partial x}(P), \frac{\partial f}{\partial y}(P), \frac{\partial f}{\partial z}(P) \right)$$

$$\bullet \frac{\partial f}{\partial x} = 2 \cdot \frac{1}{2} (xyz)^{\frac{1}{2}-1} \cdot yz = \frac{yz}{\sqrt{xyz}}$$

$$\bullet \frac{\partial f}{\partial y} = 2 \cdot \frac{1}{2} \cdot (xyz)^{\frac{1}{2}-1} \cdot xz = \frac{xz}{\sqrt{xyz}}$$

$$\bullet \frac{\partial f}{\partial z} = 2 \cdot \frac{1}{2} \cdot (xyz)^{\frac{1}{2}-1} \cdot xy = \frac{xy}{\sqrt{xyz}}$$

$$\Rightarrow \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = \left(\frac{yz}{\sqrt{xyz}}, \frac{xz}{\sqrt{xyz}}, \frac{xy}{\sqrt{xyz}} \right)$$

$$\Rightarrow \nabla f = \frac{1}{\sqrt{xyz}} \cdot (yz, xz, xy)$$

Diese Formel, einsetzen:

$$\underline{\nabla f(P)} = \nabla f(3, -4, -3) =$$

$$= \frac{1}{\sqrt{3 \cdot (-4) \cdot (-3)}} \cdot (-4 \cdot (-3), 3 \cdot (-3), 3 \cdot (-4))$$

$$= \frac{1}{\sqrt{3^2 \cdot 2^2}} \cdot (12, -6, -12)$$

$$= \frac{1}{6} \cdot (12, -6, -12) = \left(\frac{12}{6}, -\frac{6}{6}, -\frac{12}{6}\right)$$

$$= \underline{(2, -1, -2)}.$$

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2. Calcule a derivada direcional de  $f$  no ponto  $P$ , na direção do vetor  $\vec{v}$  em cada caso:

(a)  $f(x, y) = x^2 + 2xy + 3y^2$ ;  $P(2, 1)$ ;  $\vec{v} = (1, 1)$ .

(b)  $f(x, y, z) = \ln(1 + x^2 + y^2 - z^2)$ ;  $P(1, -1, 1)$ ;  $\vec{v} = (2, -2, 3)$ .

(c)  $f(x, y) = \arctan \frac{y}{x}$ ;  $P(-3, 3)$ ;  $\vec{v} = (3, 4)$ .

(c)  $\frac{\partial f}{\partial \vec{n}_1}(P) = ?$

$$\vec{n}_1 = \frac{\vec{n}}{\|\vec{n}\|} ;$$

$$\vec{n} = (3, 4) ; \quad \|\vec{n}\| = \sqrt{(3)^2 + (4)^2} = \sqrt{9 + 16} \\ = \|\vec{n}\| = 5$$

Assim,  $\vec{n}_1 = \frac{1}{\|\vec{n}\|} \cdot \vec{n} = \frac{1}{5} \cdot (3, 4) = \left(\frac{3}{5}, \frac{4}{5}\right)$

Dessa forma, obtenemos:

$$\frac{\partial f}{\partial \vec{n}_1}(P) = \nabla f(P) \cdot \vec{n}_1 ; \quad \text{onde:}$$

↑  
PRODUTO ESCALAR.

$$\nabla f = \left( \frac{\partial f}{\partial x} , \frac{\partial f}{\partial y} \right) = ?$$

$$f(x, y) = \arctan\left(\frac{y}{x}\right) .$$

$$\left[ \text{lemme de } C_1: (\arctan m)' = \frac{m'}{1+m^2} \right]$$

$$\bullet \frac{\partial f}{\partial x} = \frac{y \cdot (-1) x^{-2}}{1 + \left(\frac{y}{x}\right)^2} = \frac{-\frac{y}{x^2}}{\frac{x^2 + y^2}{x^2}} = -\frac{y}{x^2 + y^2}$$

$$\bullet \frac{\partial f}{\partial y} = \frac{x^{-1}}{1 + \left(\frac{y}{x}\right)^2} = \frac{\frac{1}{x}}{\frac{x^2 + y^2}{x^2}} = \frac{1}{x} \cdot \frac{x^2}{x^2 + y^2} = \frac{x}{x^2 + y^2}$$

Alors:

$$\begin{aligned} \nabla f &= \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = \left( -\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right) \\ &= \frac{1}{x^2 + y^2} \cdot (-y, x) \end{aligned}$$

Comme, donc:

$$\nabla f(P) = \nabla f(-3, 3) = \frac{1}{(-3)^2 + (3)^2} \cdot (-3, -3) =$$

$$= \frac{1}{18} \cdot (-3, -3) = \left(-\frac{3}{18}, -\frac{3}{18}\right)$$

$$= \underline{\underline{\left(-\frac{1}{6}, -\frac{1}{6}\right)}}$$

Por fim,

$$\underline{\underline{\frac{df}{d\vec{n}_1}}} = \nabla f(P) \cdot \vec{n}_1 = \left(-\frac{1}{6}, -\frac{1}{6}\right) \cdot \left(\frac{3}{5}, \frac{4}{5}\right) =$$

$$= -\frac{1}{6} \cdot \left(\frac{3}{5}\right) + \left(-\frac{1}{6}\right) \cdot \left(\frac{4}{5}\right) = -\frac{3}{30} - \frac{4}{30} = -\frac{7}{30}$$

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3. Prove o seguinte teorema:

**Teorema.** Sejam  $f, g$  funções de  $\mathbb{R}^m \rightarrow \mathbb{R}$  diferenciáveis e  $\alpha \in \mathbb{R}$ . Então

(i)  $\nabla(f + g) = \nabla f + \nabla g$ ;

(ii)  $\nabla(\alpha f) = \alpha \nabla f$ ;

(iii)  $\nabla(fg) = f\nabla g + g\nabla f$ .

(iv)  $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$

Exercício (iii)

(iii)  $\nabla(fg) = f \cdot \nabla g + g \cdot \nabla f$

Sejam  $f = f(x_1, x_2, \dots, x_m)$

$$g = g(x_1, x_2, \dots, x_m)$$

$$\Rightarrow (f \cdot g)(x) = f(x_1, \dots, x_m) \cdot g(x_1, \dots, x_m)$$

$$\nabla(f \cdot g) = \left( \frac{\partial}{\partial x_1}(f \cdot g), \frac{\partial}{\partial x_2}(f \cdot g), \dots, \frac{\partial}{\partial x_m}(f \cdot g) \right);$$

onde:

$$\frac{\partial}{\partial x_i}(f \cdot g) = f \cdot \frac{\partial g}{\partial x_i} + \frac{\partial f}{\partial x_i} \cdot g ;$$

$$\Rightarrow \nabla(f \cdot g) = \left( f \cdot \frac{\partial g}{\partial x_1} + \frac{\partial f}{\partial x_1} \cdot g, f \cdot \frac{\partial g}{\partial x_2} + \frac{\partial f}{\partial x_2} \cdot g, \dots \right. \\ \left. \dots, f \cdot \frac{\partial g}{\partial x_m} + \frac{\partial f}{\partial x_m} \cdot g \right) =$$

$$= \left( f \cdot \frac{\partial g}{\partial x_1}, f \cdot \frac{\partial g}{\partial x_2}, \dots, f \cdot \frac{\partial g}{\partial x_m} \right) + \\ + \left( \frac{\partial f}{\partial x_1} \cdot g, \frac{\partial f}{\partial x_2} \cdot g, \dots, \frac{\partial f}{\partial x_m} \cdot g \right)$$

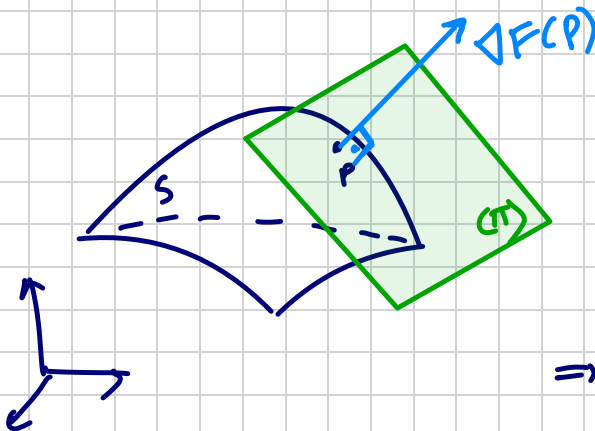
$$= f \cdot \underbrace{\left( \frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, \dots, \frac{\partial g}{\partial x_m} \right)}_{\nabla g} + \underbrace{\left( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_m} \right)}_{\nabla f} \cdot g =$$

$$= \underline{\underline{f \cdot \nabla g + g \cdot \nabla f}}$$

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Ex

6. Encontre o plano tangente ao gráfico de  $f(x, y) = xy^3 + x^2$  no ponto  $P(2, -2)$ .



Seja  $F: \mathbb{R}^3 \rightarrow \mathbb{R}$

dada por

$$F(x, y, z) = f(x, y) - z$$

$$\Rightarrow F(x, y, z) = xy^3 + x^2 - z$$

$\vec{n}$ : vetor normal ao plano  $(\pi)$  a ser obtido será  $\nabla F(P)$ .

$$\nabla F = \left( \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right)$$

$$\nabla F = (y^3 + 2x, 3xy^2, -1)$$

$$\Rightarrow \nabla F(P) = \nabla F(2, -2, \underbrace{f(2, -2)}_2) ; \text{ onde}$$

$$\underbrace{f(2, -2)} = 2 \cdot (-2)^3 + (2)^2 = -16 + 4 = \underline{-12}$$

$$\Rightarrow \nabla F(P) = \nabla F(\underbrace{2, -2, -12}_{P(2, -2, -12)})$$

Assim, teremos:

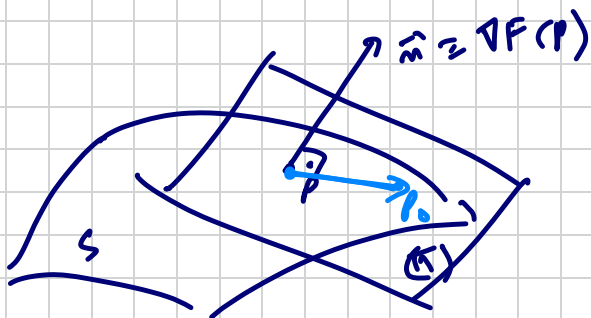
$$\begin{aligned} \nabla F(P) &= ((-2)^3 + 2 \cdot (2), 3 \cdot (2) \cdot (-2)^2, -1) \\ &= (-4, 24, -1) \end{aligned}$$

Dado  $P_0(x, y, z)$  um ponto qualquer sobre o plano  $(\pi)$  desejado, então  $\overrightarrow{P_0P} = P - P_0 \in (\pi)$ ;

onde

$$\begin{aligned} \overrightarrow{P_0P} &= P - P_0 = (x, y, z) - (2, -2, -12) \\ &= (x-2, y+2, z+12) \end{aligned}$$

Assim, a eq. do plano  $(\pi)$  será:



$$(\pi): \nabla F(P) \cdot \overrightarrow{PP_0} = 0 \quad \text{produto escalar.}$$

$$(\pi): (-4, 24, -1) \cdot (x-2, y+2, z+12) = 0$$

$$(\pi): -4 \cdot (x-2) + 24 \cdot (y+2) - 1 \cdot (z+12) = 0$$

$$(\pi): -4x + 8 + 24y + 48 - z - 12 = 0$$

$$\boxed{(\pi): -4x + 24y - z + 44 = 0}$$