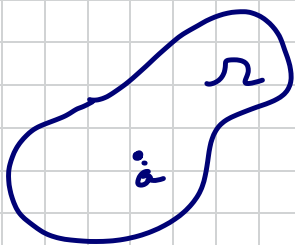


INTEGRAIS IMPROPRIAS

Da mesma forma como estado no Cálculo II, faremos para funções de duas variáveis reais um estado para integrais de funções $f: \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}$, onde f possui uma singularidade (ou seja, f não fica definida em algum ponto em Ω), ou ainda o domínio Ω é ilimitado.

1º tipo: singularidade em algum ponto em Ω .

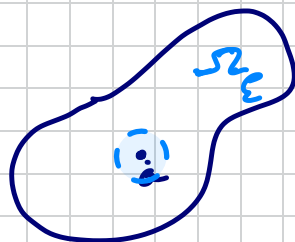


$f: \Omega \setminus \{a\} \subset \mathbb{R}^2 \rightarrow \mathbb{R}$, i.e., $\nexists f(a)$,
onde $a \in \text{int}(\Omega)$

Então, como calcular $\iint_{\Omega} f = ?$

Tomar $\varepsilon > 0$ e considere

$$\Omega_{\varepsilon} = \Omega \setminus B_{\varepsilon}(a)$$



Dessa forma, teremos

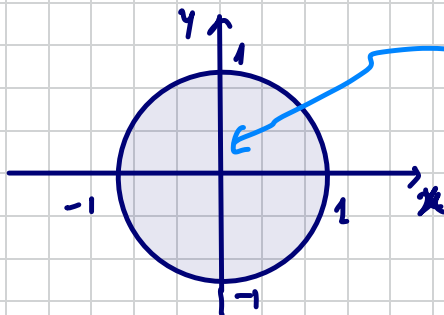
$$\iint_{\mathcal{R}} f = \lim_{\varepsilon \rightarrow 0^+} \iint_{\mathcal{R}_\varepsilon} f.$$

Se existir o limite acima, então a integral dupla $\iint_{\mathcal{R}} f$ existe, e diremos que ela converge para o valor obtido. Caso contrário diremos que a integral diverge.

Vejam os alguns exemplos:

01) Calcule $\iint_{x^2+y^2 \leq 1} \frac{dx dy}{\sqrt{x^2+y^2}}.$

Solução: A região $\mathcal{R}$ de integração será um disco de raio unitário centrado na origem do plano xy .

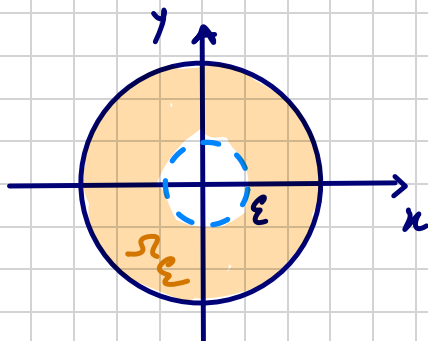


PORÉM, $(x,y) = (0,0)$ é uma SINGULARIDADE PARA f ; pois $\neq f(0,0)$.

$$f(x,y) = \frac{1}{\sqrt{x^2+y^2}}$$

Tomar a região Ω_ε , onde $0 < \varepsilon < 1$, dada por:

$$\Omega_\varepsilon = \Omega \setminus B_\varepsilon(0,0)$$



Assim, temos:

$$\iint_{\Omega} f(x,y) dx dy = \lim_{\varepsilon \rightarrow 0^+} \iint_{\Omega_\varepsilon} f(x,y) dx dy =$$

$$= \lim_{\varepsilon \rightarrow 0^+} \int_{\theta=0}^{\theta=2\pi} \int_{\rho=\varepsilon}^{\rho=1} \frac{\cancel{\rho} d\rho d\theta}{\cancel{\rho}} = \lim_{\varepsilon \rightarrow 0^+} \int_{\theta=0}^{\theta=2\pi} \left(\int_{\rho=\varepsilon}^{\rho=1} d\rho \right) d\theta$$

COORDENADAS
POLARES

$$dx dy = \rho d\rho d\theta$$

$$\rho = \sqrt{x^2 + y^2}$$

$$= \lim_{\varepsilon \rightarrow 0^+} \int_{\theta=0}^{\theta=2\pi} \rho \Big|_{\varepsilon}^1 d\theta =$$

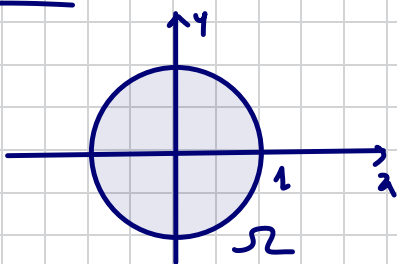
$$= \lim_{\varepsilon \rightarrow 0^+} \int_{\theta=0}^{\theta=2\pi} (1 - \varepsilon) d\theta = \lim_{\varepsilon \rightarrow 0^+} (1 - \varepsilon) \cdot \int_{\theta=0}^{\theta=2\pi} d\theta =$$

$$= \lim_{\xi \rightarrow 0^+} (1-\xi) \cdot 2\pi = 1 \cdot 2\pi = \underline{\underline{2\pi}}.$$

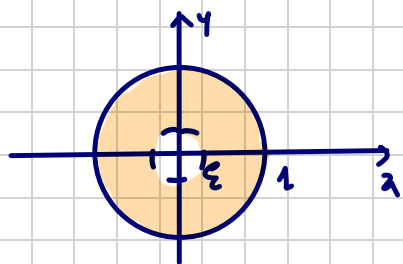
02) $\iint_{\Omega} \ln\left(\frac{1}{x^2+y^2}\right) dx dy$; onde

$$\Omega = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\}.$$

Solução:



$\Rightarrow$



$$0 < \xi < 1$$

$$\Omega_\xi = \Omega \setminus B(0, \xi).$$

Assim:

$$\iint_{\Omega} f = \lim_{\xi \rightarrow 0^+} \iint_{\Omega_\xi} f = \lim_{\xi \rightarrow 0^+} \int_{\theta=0}^{\theta=2\pi} \int_{\rho=\xi}^{\rho=1} \ln\left(\frac{1}{\rho^2}\right) \cdot \rho d\rho d\theta$$

coord.
polares

$$= \lim_{\xi \rightarrow 0^+} \int_{\theta=0}^{\theta=2\pi} \int_{\rho=\xi}^{\rho=1} \ln(\rho^{-2}) \cdot \rho d\rho d\theta =$$

$$\ln a^m = m \ln a$$

$$= \lim_{\epsilon \rightarrow 0^+} \int_{\theta=0}^{\theta=2\pi} \int_{\rho=\epsilon}^{\rho=1} -2 \cdot \ln(\rho) \cdot \rho \, d\rho \, d\theta =$$

$$= -2 \cdot \lim_{\epsilon \rightarrow 0^+} \int_{\theta=0}^{\theta=2\pi} \left(\int_{\rho=\epsilon}^{\rho=1} \ln \rho \cdot \rho \, d\rho \right) d\theta =$$

INT. FOR PARTES:

$$\int \ln \rho \cdot \rho \, d\rho = \int u \, dv = u \cdot v - \int v \, du$$

$$\left[\begin{array}{l} u = \ln \rho \Rightarrow du = \frac{1}{\rho} d\rho \\ dv = \rho \, d\rho \Rightarrow v = \int \rho \, d\rho = \frac{\rho^2}{2} \end{array} \right.$$

$$= \ln \rho \cdot \frac{\rho^2}{2} - \int \frac{\rho^2}{2} \cdot \frac{d\rho}{\rho}$$

$$= \frac{\rho^2}{2} \cdot \ln \rho - \frac{1}{2} \int \rho \, d\rho = \frac{\rho^2}{2} \cdot \ln \rho - \frac{\rho^2}{4} + C$$

$$= -2 \cdot \lim_{\epsilon \rightarrow 0^+} \int_{\theta=0}^{\theta=2\pi} \left(\frac{\rho^2}{2} \cdot \ln \rho - \frac{\rho^2}{4} \right) \bigg|_{\epsilon}^1 d\theta$$

$$= -2 \cdot \lim_{\varepsilon \rightarrow 0^+} 2\pi \cdot \frac{\varepsilon^2}{2} \left(\ln \varepsilon - \frac{1}{2} \right) \Big|_{\varepsilon}^1 =$$

$$= -4\pi \cdot \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2} \cdot \underbrace{\left(\ln 1 - \frac{1}{2} \right)}_0 - \frac{\varepsilon^2}{2} \left(\ln \varepsilon - \frac{1}{2} \right)$$

$$= -4\pi \cdot \lim_{\varepsilon \rightarrow 0^+} \left(-\frac{1}{4} - \frac{\varepsilon^2}{2} \cdot \ln \varepsilon + \frac{\varepsilon^2}{4} \right) =$$

$$= -4\pi \left[-\frac{1}{4} - \frac{1}{2} \lim_{\varepsilon \rightarrow 0^+} \underbrace{\varepsilon^2 \cdot \ln \varepsilon}_{\infty} + \lim_{\varepsilon \rightarrow 0^+} \underbrace{\frac{\varepsilon^2}{4}}_0 \right]$$

$\infty \cdot 0$ e!
 indeterminação,

$$= \pi + 2\pi \cdot \lim_{\varepsilon \rightarrow 0^+} \underbrace{\varepsilon^2 \cdot \ln \varepsilon}_{0 \cdot \infty} + 0 =$$

$$= \pi + 2\pi \cdot \lim_{\varepsilon \rightarrow 0^+} \frac{\ln \varepsilon}{\frac{1}{\varepsilon^2}} = \pi + 2\pi \cdot \lim_{\varepsilon \rightarrow 0^+} \frac{\frac{1}{\varepsilon}}{-2\varepsilon^{-3}}$$

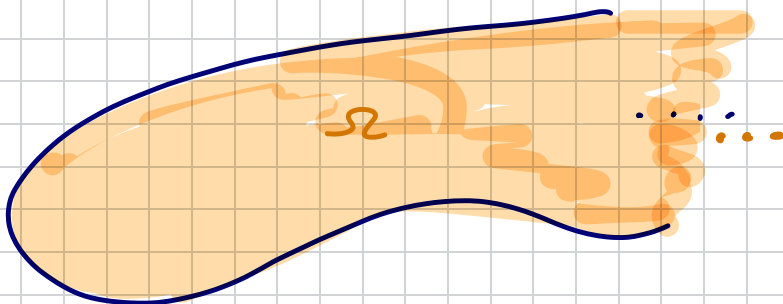
$\frac{\infty}{\infty}$ L'HOPITAL

$$= \pi + 2\pi \cdot \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\cancel{\varepsilon}} \cdot \left(-\frac{1}{2} \cdot \varepsilon^3 \right) =$$

$$= \pi + 2\pi \cdot \lim_{\varepsilon \rightarrow 0^+} \left(-\frac{1}{2} \cdot \varepsilon^2 \right) = \pi.$$

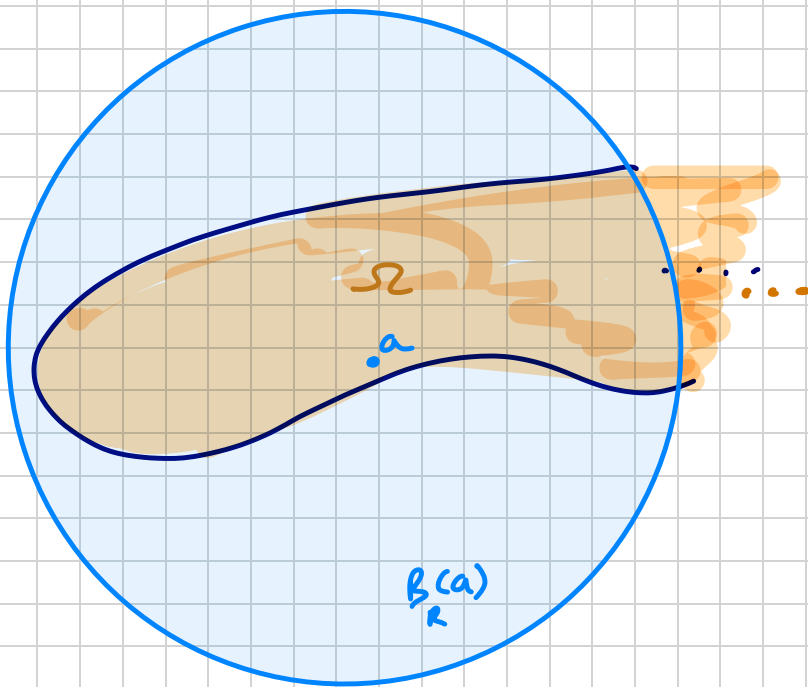
conclusão: $\iint_{x^2+y^2 \leq 1} \ln\left(\frac{1}{x^2+y^2}\right) dx dy = \pi.$

2º tipo: Quando a região Ω de integração for ilimitada.



Neste caso, tome um retângulo ou uma bola que, quando suas dimensões (largura ou raio) tendem ao infinito, cobrirá todo o Ω .

Ex.:



$$\Omega_R = \Omega \cap B_R(a) .$$

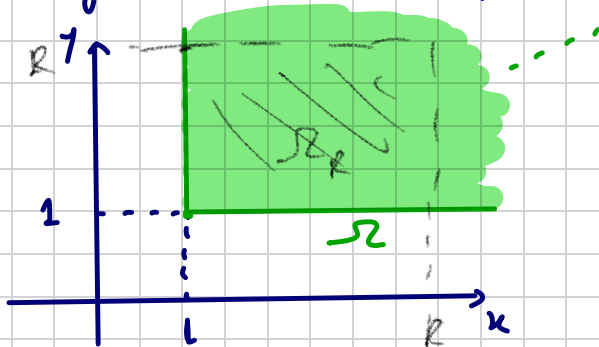
$$\text{Assim: } \iint_{\Omega} f = \lim_{R \rightarrow +\infty} \iint_{\Omega_R} f .$$

Da mesma forma que no 1º TIPO, diremos que a integral converge se existir o limite acima. Do contrário, é dita divergente.

Vejam os exemplos:

02) Calcule $\int_1^{+\infty} \int_1^{+\infty} \frac{dx dy}{x^2 y^2}$.

Solução: A região Ω é dada por:



Tomemos $\Omega_R = \Omega \cap [1, R] \times [1, R]$. Assim:

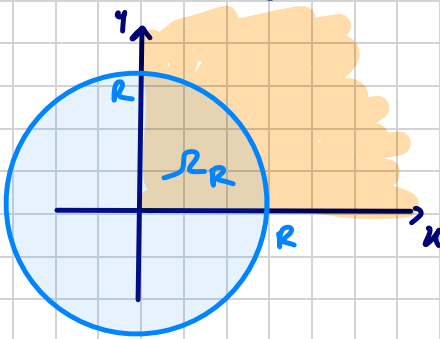
$$\int_1^{+\infty} \int_1^{+\infty} \frac{dx dy}{x^2 y^2} = \lim_{R \rightarrow +\infty} \int_{y=1}^{y=R} \int_{x=1}^{x=R} \frac{dx dy}{x^2 y^2} =$$

$$= \lim_{R \rightarrow +\infty} \int_{y=1}^{y=R} \frac{dy}{y^2} \int_{x=1}^{x=R} \frac{dx}{x^2} =$$

$$= \lim_{R \rightarrow +\infty} \int_{y=1}^{y=R} y^{-2} dy \cdot \int_{x=1}^{x=R} x^{-2} dx =$$

$$\begin{aligned}
 &= \lim_{R \rightarrow +\infty} \frac{y^{-1}}{-1} \Big|_{-1}^R \cdot \frac{x^{-1}}{-1} \Big|_1^R = \\
 &= \lim_{R \rightarrow +\infty} \left(-\frac{1}{R} + 1 \right) \cdot \left(-\frac{1}{R} + 1 \right) = 1.
 \end{aligned}$$

02) $\int_0^{+\infty} \int_0^{+\infty} e^{-x^2-y^2} dx dy = ?$



$$R_R = [0, +\infty) \times [0, +\infty) \cap B(0)_R$$

$$R = \lim_{R \rightarrow +\infty} R_R$$

Answer, theorem: (2 dimensional coordinates polar)

$$\int_0^{+\infty} \int_0^{+\infty} e^{-x^2-y^2} dx dy = \lim_{R \rightarrow +\infty} \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{\rho=0}^{\rho=R} e^{-\rho^2} \rho d\rho d\theta =$$

$$-x^2 - y^2 = -(x^2 + y^2) = -\rho^2$$

$$= \lim_{R \rightarrow +\infty} \int_{\theta=0}^{\theta=\frac{\pi}{2}} d\theta \cdot \left(-\frac{1}{2}\right) \int_{r=0}^{r=R} e^{-r^2} (-2r) dr =$$

$$\int e^u du$$

$$u = -r^2$$

$$du = -2r dr$$

$$= -\frac{1}{2} \cdot \lim_{R \rightarrow +\infty} \theta \bigg|_0^{\frac{\pi}{2}} \cdot e^{-r^2} \bigg|_{r=0}^{r=R} =$$

$$= -\frac{1}{2} \cdot \frac{\pi}{2} \cdot \lim_{R \rightarrow +\infty} (e^{-R^2} - 1) =$$

$$= -\frac{\pi}{4} \cdot \lim_{R \rightarrow +\infty} \left(\frac{1}{e^R} - 1 \right) = -\frac{\pi}{4} \cdot (-1) = \frac{\pi}{4}$$

LISTA 03

13. Calcule as integrais impróprias abaixo.

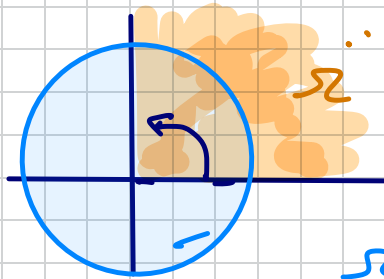
(a) $\int_0^\infty \int_0^\infty e^{-x-y} dx dy$

(b) $\int_0^\infty \int_0^\infty x^2 e^{-x^2-y^2} dx dy$

(c) $\iint_{x^2+y^2 \leq 1} \frac{x^2 dx dy}{(x^2+y^2)^{\frac{7}{4}}}$

(d) $\iint_{x^2+y^2 \leq 1} \ln \sqrt{x^2+y^2} dx dy$

$$(b) \int_0^{+\infty} \int_0^{+\infty} x^2 \cdot e^{-x^2-y^2} dx dy = \lim_{R \rightarrow +\infty} \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{\rho=0}^{\rho=R} (\rho \cos \theta)^2 \cdot e^{-\rho^2} \cdot \rho d\rho d\theta \quad \text{---}$$



$$\Sigma_R = \Sigma \cap B(0)_R.$$

$$x = \rho \cos \theta$$

$$dx dy = \rho d\rho d\theta. \quad x^2 + y^2 = \rho^2$$

$$\text{---} \lim_{R \rightarrow +\infty} \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{\rho=0}^{\rho=R} \rho^2 \cos^2 \theta \cdot e^{-\rho^2} \rho d\rho d\theta =$$

$$= \lim_{R \rightarrow +\infty} \int_{\theta=0}^{\theta=\frac{\pi}{2}} \cos^2 \theta d\theta \int_{\rho=0}^{\rho=R} e^{-\rho^2} \cdot \rho^3 d\rho = \dots$$



$$\cos^2 \theta = \frac{1 + \cos 2\theta}{2}$$

(rema ... concluir)
 ↓
 indiguen por partes...