

RESOLUÇÃO DE EXERCÍCIOS DA LISTA 05.

2. De cada função vetorial a seguir, obtenha a matriz Jacobiana:

(a) $f(x, y) = (e^{x^2+y^2}, 2x^2y + 3y^2, \sqrt{x^2 + y^2})$

(b) $f(x, y, z) = (\underbrace{x^2 + y^2 + z^2}_{f_1}, \underbrace{xyz}_{f_2}, \underbrace{\cos xy}_{f_3}, \underbrace{x^2 - yz}_{f_4})$

(b) $f: \mathbb{R}^3 \rightarrow \mathbb{R}^4$

$[d_a f] = (f_{ij})_{4 \times 3}$, onde $f_{ij} = \frac{\partial f_i}{\partial x_j}$

$(x_1 = x; x_2 = y; x_3 = z)$

• $f_1(x, y, z) = x^2 + y^2 + z^2$

$\frac{\partial f_1}{\partial x} = 2x$; $\frac{\partial f_1}{\partial y} = 2y$; $\frac{\partial f_1}{\partial z} = 2z$

• $f_2(x, y, z) = xyz$

$\frac{\partial f_2}{\partial x} = yz$; $\frac{\partial f_2}{\partial y} = xz$; $\frac{\partial f_2}{\partial z} = xy$

• $f_3(x, y, z) = \cos xy$

$\frac{\partial f_3}{\partial x} = -y \cdot \sin xy$; $\frac{\partial f_3}{\partial y} = -x \sin xy$; $\frac{\partial f_3}{\partial z} = 0$

$$\bullet f_4(x, y, z) = x^2 - yz.$$

$$\frac{\partial f_4}{\partial x} = 2x; \quad \frac{\partial f_4}{\partial y} = -z; \quad \frac{\partial f_4}{\partial z} = -y$$

Dimo, otteniamo:

$$\frac{df}{dx} = (f_{ij})_{4 \times 3} = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \\ f_{41} & f_{42} & f_{43} \end{bmatrix} =$$

$$= \begin{bmatrix} 2x & 2y & 2z \\ yz & xz & xy \\ -y \cdot \text{sen} x y & -x \cdot \text{sen} x y & 0 \\ 2x & -z & -y \end{bmatrix}_a$$



3. Seja $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ dada por

$$f(x, y) = \begin{cases} \frac{3x^2y^2}{x^4 + y^4}, & \text{se } (x, y) \neq (0, 0) \\ 0, & \text{se } (x, y) = (0, 0) \end{cases}.$$

Mostre que $\frac{\partial}{\partial x} f(0, 0)$ e $\frac{\partial}{\partial y} f(0, 0)$ existem, mas que f não é diferenciável na origem.

$$\begin{aligned} \bullet \quad \frac{\partial f}{\partial x}(0, 0) &= \lim_{h \rightarrow 0} \frac{f(0+h, 0) - f(0, 0)}{h} = \\ &= \lim_{h \rightarrow 0} \frac{f(h, 0) - 0}{h} = \lim_{h \rightarrow 0} \frac{\frac{3 \cdot h^2 \cdot 0^2}{h^4 + 0^4} - 0}{h} \\ &= \lim_{h \rightarrow 0} \frac{0}{h} = 0 \quad \Rightarrow \boxed{\frac{\partial f}{\partial x}(0, 0) = 0} \end{aligned}$$

$$\begin{aligned} \bullet \quad \frac{\partial f}{\partial y}(0, 0) &= \lim_{h \rightarrow 0} \frac{f(0, 0+h) - f(0, 0)}{h} = \\ &= \lim_{h \rightarrow 0} \frac{f(0, h) - 0}{h} = \lim_{h \rightarrow 0} \frac{\frac{3 \cdot 0^2 \cdot h^2}{0^4 + h^4} - 0}{h} \\ &= \lim_{h \rightarrow 0} \frac{0}{h} = 0 \quad \Rightarrow \boxed{\frac{\partial f}{\partial y}(0, 0) = 0} \end{aligned}$$

Descreva do seguinte teorema:

teor.: Se $f: \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ é diferenciável em um ponto $a \in \text{int}(\Omega)$, então f é contínua nesse ponto.

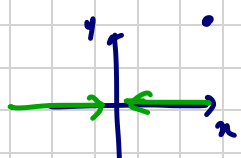
Logo, por contraposição, se f não for contínua em um ponto, então f não será diferenciável nesse ponto.

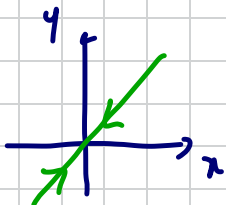
Então, a parte final do problema resume-se a verificar que f não é contínua na origem.

$$f(x, y) = \begin{cases} \frac{3x^2y^2}{x^4 + y^4}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

AF.: f não é contínua na origem.

De fato, note que:


$$\begin{aligned} \lim_{\substack{(x,y) \rightarrow (0,0) \\ y=0}} f(x,y) &= \lim_{\substack{(x,y) \rightarrow (0,0) \\ y=0}} \frac{3x^2 \cdot 0^2}{x^4 + 0^4} = \\ &= \lim_{\substack{(x,y) \rightarrow (0,0) \\ y=0}} \frac{0}{x^4} = 0 // \end{aligned}$$



$$\lim_{\substack{(x,y) \rightarrow (0,0) \\ y=x}} f(x,y) = \lim_{\substack{(x,y) \rightarrow (0,0) \\ y=x}} \frac{3x^2 \cdot x^2}{x^4 + x^4} =$$

$$= \lim_{\substack{(x,y) \rightarrow (0,0) \\ y=x}} \frac{\cancel{3x^4}}{\cancel{2x^4}} = \frac{3}{2}$$

$$\Rightarrow \lim_{\substack{(x,y) \rightarrow (0,0) \\ y=0}} f(x,y) = 0 \neq \frac{3}{2} = \lim_{\substack{(x,y) \rightarrow (0,0) \\ y=x}} f(x,y).$$

Logo, $\nexists \lim_{(x,y) \rightarrow (0,0)} f(x,y)$; e, portanto, f não é
contínua na origem.

5. Calcule as diferenciais totais de cada função a seguir:

(a) $z = \frac{x}{\sqrt{x^2 + y^2}}$

(b) $z = \ln(xy + y^2)$

(c) $z = \arctan \frac{x+y}{1-xy}$

(d) $z = \frac{ye^x}{\sqrt{x^2 + y^2}}$

(e) $z = \arcsin(x\sqrt{1-y^2} + y\sqrt{1-x^2})$

(f) $z = \frac{x \sin y}{\cos(xy)}$

(c) $f(x,y) = \arctan \frac{x+y}{1-xy}$

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \quad ; \quad \text{onde:}$$

$$\frac{\partial f}{\partial x} = \frac{2}{2x} \left(\arctan \frac{x+y}{1-xy} \right) = \frac{\frac{2}{2x} \left(\frac{x+y}{1-xy} \right)}{1 + \left(\frac{x+y}{1-xy} \right)^2} =$$

$$(\arctan u)' = \frac{u'}{1+u^2}$$

$$\left(\frac{u}{v} \right)' = \frac{v \cdot u' - u \cdot v'}{v^2}$$

$$= \frac{\frac{(1-xy) \cdot 1 - (x+y) \cdot (-y)}{(1-xy)^2}}{1 + \frac{(x+y)^2}{(1-xy)^2}} =$$

$$= \frac{\frac{1 - \cancel{xy} + \cancel{xy} + y^2}{(1 - \cancel{xy})^2}}{\frac{(1-xy)^2 + (x+y)^2}{(\cancel{1-xy})^2}} = \frac{1 + y^2}{(1-xy)^2 + (x+y)^2}$$

$$= \frac{1 + y^2}{1 - \cancel{2xy} + x^2 y^2 + x^2 + \cancel{2xy} + y^2} =$$

$$= \frac{1+y^2}{1+x^2y^2+x^2+y^2}$$

$$\rightarrow \boxed{\frac{\partial f}{\partial x} = \frac{1+y^2}{1+x^2y^2+x^2+y^2}}$$

$$\cdot \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \left(\arctan \frac{x+y}{1-xy} \right) = \frac{1+x^2}{1+x^2y^2+x^2+y^2}$$

Satz von, alternier:

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

$$\Rightarrow df = \frac{1+y^2}{1+x^2y^2+x^2+y^2} \cdot dx + \frac{1+x^2}{1+x^2y^2+x^2+y^2} \cdot dy$$



6. Calcule cada derivada parcial indicada, usando a regra da cadeia.

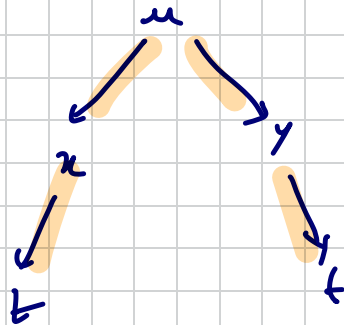
(a) $u = \ln xy + y^2$, onde $x = e^t$ e $y = e^{-t}$. Obter $\frac{\partial u}{\partial t}$.

$$u = \ln xy + y^2 .$$

$$x = e^t ; y = e^{-t}$$

$$\frac{\partial u}{\partial t} = ?$$

.



$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t} ; \text{ onde:}$$

$$\frac{\partial u}{\partial x} = \frac{1}{xy} + 0 = \frac{1}{x}$$

$$\textcircled{(\ln r)' = \frac{r'}{r}}$$

$$\frac{\partial u}{\partial y} = \frac{x}{xy} + 2y$$

$$= \frac{1}{y} + 2y$$

$$\frac{\partial x}{\partial t} = \frac{d}{dt}(e^t) = e^t ; \quad \frac{\partial y}{\partial t} = \frac{d}{dt}(e^{-t}) = -e^{-t}$$

Ansinn, ablesen:

$$\frac{\partial u}{\partial t} = \frac{1}{2} \cdot e^t + \left(\frac{1}{4} + 2y\right) \cdot (-e^{-t})$$

$$= \frac{1}{e^t} \cdot \cancel{e^t} + \left(\frac{1}{e^{-t}} + 2 \cdot \bar{e}^t\right) \cdot (-\bar{e}^t)$$

$$= 1 + -\frac{\cancel{e^t}}{\cancel{e^{-t}}} - 2\bar{e}^t \cdot \bar{e}^t$$

$$= \cancel{1} - \cancel{1} - 2\bar{e}^{-2t} = -2\bar{e}^{-2t}$$

$$\Rightarrow \boxed{\frac{\partial u}{\partial t} = -2\bar{e}^{-2t}}$$

7. Se $u = f(x, y)$ é uma função diferenciável de x e y com $x = \rho \cos \theta$ e $y = \rho \sin \theta$, mostre que

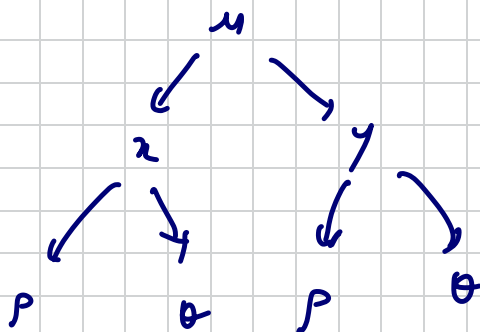
$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \rho} \cos \theta - \frac{\partial u}{\partial \theta} \frac{\sin \theta}{\rho}$$

e

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \rho} \sin \theta + \frac{\partial u}{\partial \theta} \frac{\cos \theta}{\rho}$$

obs: chamam-se
coordenadas
polares

$$\begin{cases} x = \rho \cos \theta \\ y = \rho \sin \theta \end{cases}$$



Note que:

$$\frac{\partial u}{\partial \rho} = \frac{\partial u}{\partial x} \cdot \underbrace{\frac{\partial x}{\partial \rho}}_{\cos \theta} + \frac{\partial u}{\partial y} \cdot \underbrace{\frac{\partial y}{\partial \rho}}_{\sin \theta}$$

$$\rightarrow \boxed{\frac{\partial u}{\partial \rho} = \frac{\partial u}{\partial x} \cdot \cos \theta + \frac{\partial u}{\partial y} \cdot \sin \theta} \quad (I)$$

Além disso:

$$x = \rho \cos \theta$$

$$y = \rho \sin \theta$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x} \cdot \underbrace{\frac{\partial x}{\partial \theta}}_{-\sin \theta} + \frac{\partial u}{\partial y} \cdot \underbrace{\frac{\partial y}{\partial \theta}}_{\rho \cos \theta}$$

$$\Rightarrow \boxed{\frac{\partial u}{\partial \theta} = -p \sin \theta \cdot \frac{\partial u}{\partial x} + p \cos \theta \cdot \frac{\partial u}{\partial y}} \quad (II)$$

Isolando $\frac{\partial u}{\partial y}$ de (II) temos:

$$\frac{\partial u}{\partial y} = \frac{1}{p \cos \theta} \cdot \frac{\partial u}{\partial \theta} + \frac{\cancel{p} \sin \theta}{\cancel{p} \cos \theta} \cdot \frac{\partial u}{\partial x}$$

Derivando para (I), obtemos:

$$\frac{\partial u}{\partial p} = \frac{\partial u}{\partial x} \cdot \cos \theta + \left(\frac{1}{p \cos \theta} \cdot \frac{\partial u}{\partial \theta} + \frac{\sin \theta}{\cos \theta} \cdot \frac{\partial u}{\partial x} \right) \cdot \sin \theta$$

$$= \underbrace{\frac{\partial u}{\partial x} \cos \theta}_{=1} + \frac{\sin \theta}{p \cos \theta} \cdot \frac{\partial u}{\partial \theta} + \underbrace{\frac{\sin^2 \theta}{\cos \theta} \cdot \frac{\partial u}{\partial x}}_{=0}$$

$$= \left(\cos \theta + \frac{\sin^2 \theta}{\cos \theta} \right) \frac{\partial u}{\partial x} + \frac{\sin \theta}{p \cdot \cos \theta} \cdot \frac{\partial u}{\partial \theta}$$

$$= \frac{\overbrace{\cos^2 \theta + \sin^2 \theta}^{=1}}{\cos \theta} \cdot \frac{\partial u}{\partial x} + \frac{\sin \theta}{p \cos \theta} \cdot \frac{\partial u}{\partial \theta}$$

$$= \frac{1}{\cos \theta} \cdot \frac{\partial u}{\partial x} + \frac{\sin \theta}{\rho \cos \theta} \cdot \frac{\partial u}{\partial \theta}.$$

$$\Rightarrow \frac{\partial u}{\partial \rho} = \frac{1}{\cos \theta} \cdot \frac{\partial u}{\partial x} + \frac{\sin \theta}{\rho \cos \theta} \cdot \frac{\partial u}{\partial \theta}.$$

Por fim, isolando $\frac{\partial u}{\partial x}$, vem:

$$\Rightarrow \frac{\partial u}{\partial x} = \cos \theta \cdot \frac{\partial u}{\partial \rho} - \cancel{\cos \theta} \cdot \frac{\cancel{\sin \theta}}{\cancel{\rho} \cos \theta} \cdot \frac{\partial u}{\partial \theta}$$

$$\Rightarrow \boxed{\frac{\partial u}{\partial x} = \cos \theta \cdot \frac{\partial u}{\partial \rho} - \frac{\sin \theta}{\rho} \cdot \frac{\partial u}{\partial \theta}};$$

que prova a 1ª igualdade desejada.

Derivando esse resultado para (I), obtemos:

$$\frac{\partial u}{\partial \rho} = \frac{\partial u}{\partial x} \cos \theta + \frac{\partial u}{\partial y} \sin \theta.$$

$$\Rightarrow \frac{\partial u}{\partial y} = \frac{1}{\sin \theta} \cdot \frac{\partial u}{\partial \rho} - \frac{1}{\sin \theta} \cdot \cos \theta \cdot \frac{\partial u}{\partial x}$$

$$= \frac{1}{\sin \theta} \cdot \frac{\partial M}{\partial p} - \frac{\cos \theta}{\sin \theta} \cdot \left(\cos \theta \cdot \frac{\partial M}{\partial p} - \frac{\sin \theta}{p} \cdot \frac{\partial M}{\partial \theta} \right)$$

$$= \frac{1}{\sin \theta} \cdot \frac{\partial M}{\partial p} - \frac{\cos^2 \theta}{\sin \theta} \cdot \frac{\partial M}{\partial p} + \frac{\cos \theta \cdot \cancel{\sin \theta}}{p \cdot \cancel{\sin \theta}} \cdot \frac{\partial M}{\partial \theta}$$

$$= \left(\frac{1}{\sin \theta} - \frac{\cos^2 \theta}{\sin \theta} \right) \frac{\partial M}{\partial p} + \frac{\cos \theta}{p} \cdot \frac{\partial M}{\partial \theta}$$

$$= \frac{\overbrace{1 - \cos^2 \theta}^{\sin^2 \theta}}{\sin \theta} \cdot \frac{\partial M}{\partial p} + \frac{\cos \theta}{p} \cdot \frac{\partial M}{\partial \theta}$$

$$= \frac{\cancel{\sin^2 \theta}}{\cancel{\sin \theta}} \cdot \frac{\partial M}{\partial p} + \frac{\cos \theta}{p} \cdot \frac{\partial M}{\partial \theta}$$

$$\Rightarrow \boxed{\frac{\partial M}{\partial y} = \sin \theta \cdot \frac{\partial M}{\partial p} + \frac{\cos \theta}{p} \frac{\partial M}{\partial \theta}}$$