

TEOREMA (REGRAS DA CADEIA) Sejam $f: \Omega \subset \mathbb{R}^m \rightarrow \mathbb{R}^n$
 uma função diferenciável em $a \in \text{int}(\Omega)$, $g: \mathbb{R}^n \rightarrow \mathbb{R}^k$
 diferenciável em $b = f(a)$; então $g \circ f: \Omega \subset \mathbb{R}^m \rightarrow \mathbb{R}^k$
 também é diferenciável no ponto a , com

$$(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$$

Em notação de diferenciação:

$$\begin{bmatrix} d \\ a \end{bmatrix}_{g \circ f} = \begin{bmatrix} d \\ f(a) \end{bmatrix}_g \cdot \begin{bmatrix} d \\ a \end{bmatrix}_f$$

$k \times m$ $k \times n$ $n \times m$

PRODUTO MATRICIAL

A demonstração desse teorema foge de um curso tradicional de cálculo.

Vejam os um exemplo de aplicação:

$$f: \mathbb{R}^3 \rightarrow \mathbb{R}^2, \quad f(x, y, z) = (xy, x^2y^2 + z^2)$$

$$g: \mathbb{R}^2 \rightarrow \mathbb{R}^4; \quad g(x, y) = (3x - 2y, x^2, xy^2, y)$$

Obter $d_{(1,2,-1)}(g \circ f)$.

Solução: Uma forma de fazer seria obter primeiro a composição $g \circ f$, denotando-a por $h = g \circ f$.

Por fim, calcule $d_a h$. (a matriz jacobiana de h)

Vamos, no entanto, por outro caminho: usar a regra da cadeia:

$$\left[d_a (g \circ f) \right]_{4 \times 3} = \left[d_{f(a)} g \right]_{4 \times 2} \cdot \left[d_a f \right]_{2 \times 3},$$

$$\text{onde } g = (\underbrace{3x - 2y}_{g_1}, \underbrace{x^2}_{g_2}, \underbrace{xy^2}_{g_3}, \underbrace{y}_{g_4}), \text{ e}$$

$$f = (\underbrace{xy}_{f_1}, \underbrace{x^2 + y^2 + z^2}_{f_2})$$

Ans:im:

$$\underbrace{\left[\frac{df}{da} \right]}_{2 \times 3} = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \end{bmatrix}_a = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{bmatrix}_a$$

$$= \begin{bmatrix} y & x & 0 \\ 2x & 2y & 2z \end{bmatrix}_{a=(1,2,-1)} = \begin{bmatrix} 2 & 1 & 0 \\ 2 \cdot 1 & 2 \cdot 2 & 2 \cdot (-1) \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 0 \\ 2 & 4 & -2 \end{bmatrix}$$

$$\left[\frac{dg}{f(a)} \right]_{4 \times 2} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \\ g_{31} & g_{32} \\ g_{41} & g_{42} \end{bmatrix}_{f(a)} = \begin{bmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \\ \frac{\partial g_3}{\partial x} & \frac{\partial g_3}{\partial y} \\ \frac{\partial g_4}{\partial x} & \frac{\partial g_4}{\partial y} \end{bmatrix}_{f(a)}$$

$$= \begin{bmatrix} 3 & -2 \\ 2x & 0 \\ y^2 & 2xy \\ 0 & 1 \end{bmatrix} f(a)$$

$$= \begin{bmatrix} 3 & -2 \\ 2 \cdot 2 & 0 \\ (6)^2 & 2 \cdot 2 \cdot 6 \\ 0 & 1 \end{bmatrix} =$$

$$g_1 = 3x - 2y$$

$$g_2 = x^2$$

$$g_3 = xy^2$$

$$g_4 = y$$

$$a = (1, 2, -1); f(x, y) = (xy, x^2 + y^2 + z^2)$$

$$f(1, 2, -1) = (1 \cdot 2, 1^2 + (2)^2 + (-1)^2) = (2, 6)$$

$$= \begin{bmatrix} 3 & -2 \\ 4 & 0 \\ 36 & 24 \\ 0 & 1 \end{bmatrix}$$

Answer:

$$\underline{d_a g \circ f} = \underline{d_{f(a)} g} \cdot \underline{d_a f} = \begin{bmatrix} 3 & -2 \\ 4 & 0 \\ 36 & 24 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 & 0 \\ 2 & 4 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \cdot 2 - 2 \cdot 2 & 3 \cdot 1 - 2 \cdot 4 & 3 \cdot 0 - 2 \cdot (-2) \\ 4 \cdot 2 + 0 \cdot 2 & 4 \cdot 1 + 0 \cdot 4 & 4 \cdot 0 + 0 \cdot (-2) \\ 36 \cdot 2 + 24 \cdot 2 & 36 \cdot 1 + 24 \cdot 4 & 36 \cdot 0 + 24 \cdot (-2) \\ 0 \cdot 2 + 1 \cdot 2 & 0 \cdot 1 + 1 \cdot 4 & 0 \cdot 0 + 1 \cdot (-2) \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -5 & 4 \\ 8 & 4 & 0 \\ 120 & 132 & -48 \\ 2 & 4 & -2 \end{bmatrix}_{4 \times 3}$$

Quando as funções forem escalares, a regra da cadeia será mais simples, c.f.o seguinte resultado:

TEOREMA (Regra da cadeia para funções escalares)

Sejam $u = f(x, y)$ uma função diferenciável a duas variáveis reais, onde $x = x(s, t)$;

$y = y(s, t)$, com $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial y}{\partial s}$, $\frac{\partial y}{\partial t}$, $\frac{\partial x}{\partial s}$

e $\frac{\partial x}{\partial t}$ contínuas, s e t diferenciáveis. Então:

$$\begin{cases} \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s} \\ \frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t} \end{cases} \quad ; \quad e$$

DEMONSTRAR Como $u = f(x, y)$ é diferenciável, então, o incremento Δu é expresso por:

$$\Delta u = \frac{\partial u}{\partial x} \cdot \Delta x + \frac{\partial u}{\partial y} \cdot \Delta y + \varepsilon_1 \cdot \Delta x + \varepsilon_2 \cdot \Delta y,$$

com $\varepsilon_1, \varepsilon_2 \rightarrow 0$
 $(\Delta x, \Delta y) \rightarrow (0, 0)$

Vamos calcular $\frac{\partial u}{\partial x}$; mantendo, então, t fixo (constante)

Temos, assim, os incrementos:

$$\Delta x = x(s + \Delta s, t) - x(s, t)$$

$$\text{e } \Delta y = y(s + \Delta s, t) - y(s, t)$$

Assim:

$$\begin{aligned} \Delta u &= \frac{\partial u}{\partial x} \Delta x + \frac{\partial u}{\partial y} \Delta y + \varepsilon_1 \cdot \Delta x + \varepsilon_2 \cdot \Delta y \\ &= \frac{\partial u}{\partial x} \cdot (x(s + \Delta s, t) - x(s, t)) + \end{aligned}$$

$$+ \frac{\partial u}{\partial y} \cdot (y(n+\Delta y, t) - y(n, t)) + \xi_1 \Delta x + \xi_2 \Delta y$$

Dividindo por $\Delta x \neq 0$, vem:

$$\frac{\Delta u}{\Delta x} = \frac{\partial u}{\partial x} \cdot \frac{x(n+\Delta x, t) - x(n, t)}{\Delta x} +$$

$$+ \frac{\partial u}{\partial y} \cdot \frac{y(n+\Delta y, t) - y(n, t)}{\Delta y} + \xi_1 \frac{\Delta x}{\Delta x} + \xi_2 \frac{\Delta y}{\Delta x}$$

Tomando o limite com $\Delta x \rightarrow 0$, vamos obter:

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x} = \frac{\partial u}{\partial x} \cdot \lim_{\Delta x \rightarrow 0} \frac{x(n+\Delta x, t) - x(n, t)}{\Delta x} + \frac{\partial u}{\partial y}$$

$$\frac{\partial u}{\partial y}$$

$$+ \frac{\partial u}{\partial y} \cdot \lim_{\Delta y \rightarrow 0} \frac{y(n+\Delta y, t) - y(n, t)}{\Delta y} + \xi_2 \cdot \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$$

$$+ \xi_1 \cdot \lim_{\Delta x \rightarrow 0} \frac{\Delta x}{\Delta x} + \xi_2 \cdot \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$$

e então, obtemos:

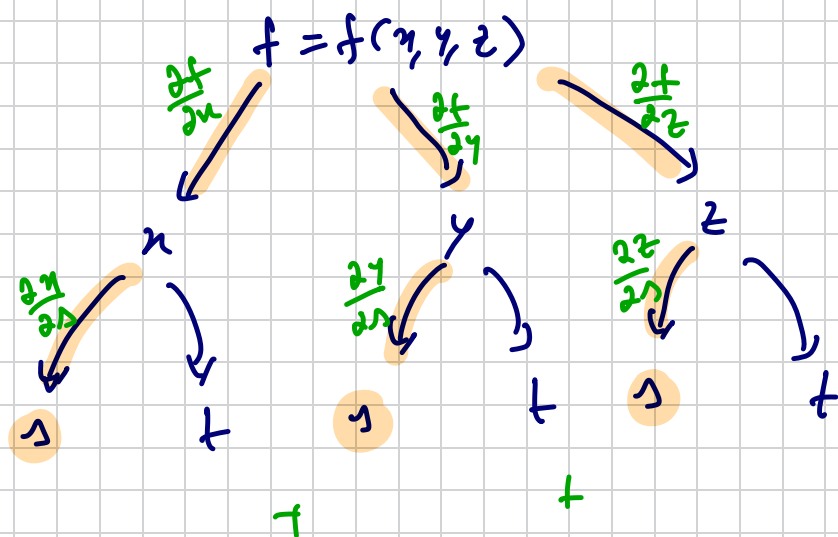
$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial x}}$$

A outra igualdade segue analogamente, mantendo s fixo e variando t .

□

Obs.: Este resultado se estende para funções de $\mathbb{R}^m$ em $\mathbb{R}$.

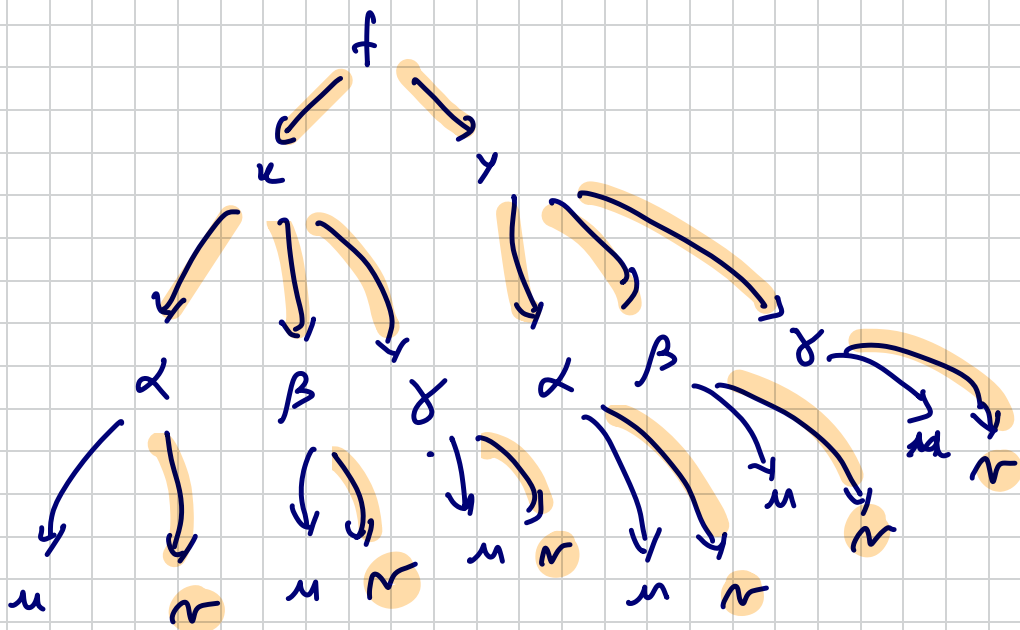
Obs.: Um esquema prático para obter a regra da cadeia (para funções de $\mathbb{R}^m$ em $\mathbb{R}$) é usar o seguinte esquema ramificado:



$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial s}.$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial t}$$

outras ramificações:



Neste caso, por exemplo:

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x} \left[\frac{\partial x}{\partial \alpha} \cdot \frac{\partial \alpha}{\partial v} + \frac{\partial x}{\partial \beta} \cdot \frac{\partial \beta}{\partial v} + \frac{\partial x}{\partial \gamma} \cdot \frac{\partial \gamma}{\partial v} \right] + \frac{\partial f}{\partial y} \left[\frac{\partial y}{\partial \alpha} \cdot \frac{\partial \alpha}{\partial v} + \frac{\partial y}{\partial \beta} \cdot \frac{\partial \beta}{\partial v} + \frac{\partial y}{\partial \gamma} \cdot \frac{\partial \gamma}{\partial v} \right]$$

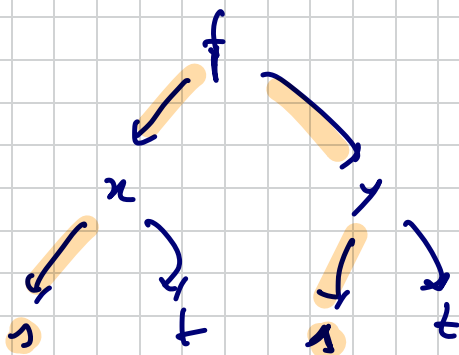
Vejaamos um exemplo:

ex.: Seja $f(x, y) = \sin(x + y^2)$, com

$$x = 1 + t \quad \text{e} \quad y = 1 + t^2. \quad \text{Determine } \frac{\partial f}{\partial t} \text{ e}$$

$$\frac{\partial f}{\partial t}.$$

Solução:



$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t}, \quad \text{onde:}$$

$$\frac{\partial f}{\partial x} = \cos(x + y^2) \cdot 1;$$

$$\frac{\partial f}{\partial y} = \cos(x + y^2) \cdot 2y = 2y \cdot \cos(x + y^2)$$

$$\frac{\partial x}{\partial t} = 1; \quad \frac{\partial y}{\partial t} = 2t.$$

Answer:

$$\frac{\partial f}{\partial \lambda} = \cos(x+y^2) \cdot t + 2y \cdot \cos(x+y^2) \cdot 2\lambda =$$

$$\begin{aligned} x &= \lambda t \\ y &= \lambda^2 + t^2 \end{aligned}$$

$$= \cos(\lambda t + (\lambda^2 + t^2)^2) \cdot t + 4 \cdot (\lambda^2 + t^2) \cdot \lambda \cdot \cos(\lambda t + (\lambda^2 + t^2)^2)$$

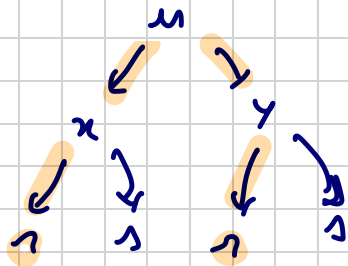
$$= \underline{\underline{\left(t + 4\lambda(\lambda^2 + t^2) \right) \cdot \cos(\lambda t + (\lambda^2 + t^2)^2)}}$$

02) (Leithold, vol 2, 3rd ed, pag 956, exercise 07)

$$u = x^2 + xy \quad ; \quad x = \lambda^2 + \lambda^2 \quad ; \quad y = 3\lambda - 2\lambda$$

$$\text{Given } \frac{\partial u}{\partial \lambda} \text{ and } \frac{\partial u}{\partial \lambda}$$

Solution:



$$\bullet \frac{\partial u}{\partial \lambda} = \underbrace{\frac{\partial u}{\partial x}} \cdot \frac{\partial x}{\partial \lambda} + \underbrace{\frac{\partial u}{\partial y}} \cdot \frac{\partial y}{\partial \lambda} =$$

$$= \underbrace{(2x+y)}_{\text{green}} \cdot \underbrace{2x}_{\text{green}} + \underbrace{x}_{\text{green}} \cdot \underbrace{3}_{\text{green}}$$

$$= 41x + 21y + 3x$$

$$= 4x \cdot (\underbrace{x^2+y^2}_{\text{blue}}) + 2x \cdot (\underbrace{3x-2y}_{\text{blue}}) + 3 \cdot (\underbrace{x^2+y^2}_{\text{blue}})$$

$$= 4x^3 + 4x \cdot \underbrace{x^2}_{\text{blue}} + \underbrace{6x^2}_{\text{blue}} - \underbrace{4xy}_{\text{blue}} + \underbrace{3x^2}_{\text{blue}} + \underbrace{3y^2}_{\text{blue}}$$

$$= 4x^3 + 9x^2 + 4x \cdot x^2 - 4xy + 3x^2$$

$$\bullet \frac{\partial u}{\partial x} = \underbrace{\frac{\partial u}{\partial x}}_{\text{green}} \cdot \underbrace{\frac{\partial y}{\partial x}}_{\text{green}} + \underbrace{\frac{\partial u}{\partial y}}_{\text{green}} \cdot \underbrace{\frac{\partial y}{\partial x}}_{\text{green}}$$

$$= (2x+y) \cdot 2x + x \cdot (-2)$$

$$= 4x^2 + 2xy - 2x$$

$$= \dots$$

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