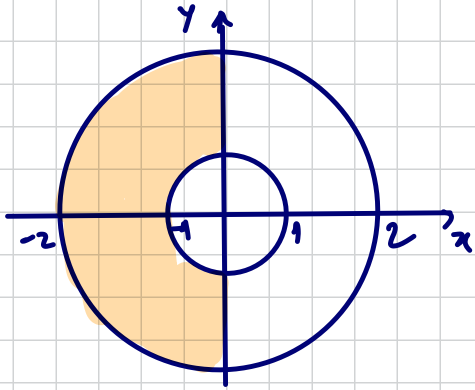
CÁLCULO 3, TURMA T3.

$\boxed{20/07/26 - \text{AULA } 46}$

AULA DE EXERCÍCIOS:

LISTA 09:

3. Calcule $\displaystyle\iint_{\Omega}(x+y)\,dxdy$, onde Ω é a região que está à esquerda do eixo y e entre as circunferências $x^2+y^2=1$ e $x^2+y^2=4$.



$1 \leq \rho \leq 2$

$\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\cdot\operatorname{sen}\theta \end{cases}$$

$dx\,dy = \rho\,d\rho\,d\theta$

$$\iint_{\Omega}(x+y)\,dx\,dy = \int_{\theta=\frac{\pi}{2}}^{\theta=\frac{3\pi}{2}}\int_{\rho=1}^{\rho=2}(\rho\cos\theta+\rho\operatorname{sen}\theta)\,\rho\,d\rho\,d\theta =$$

$$= \int_{\theta=\frac{\pi}{2}}^{\theta=\frac{3\pi}{2}}\int_{\rho=1}^{\rho=2}(\cos\theta+\operatorname{sen}\theta)\cdot\rho^2\,d\rho\,d\theta =$$

$$= \int_{\theta=\frac{\pi}{2}}^{\theta=\frac{3\pi}{2}}(\cos\theta+\operatorname{sen}\theta)\,d\theta \cdot \int_{\rho=1}^{\rho=2}\rho^2\,d\rho =$$

$$(\operatorname{sen}\theta - \cos\theta)\Big|_{\theta=\frac{\pi}{2}}^{\theta=\frac{3\pi}{2}} \cdot \frac{\rho^3}{3}\Big|_{\rho=1}^{\rho=2} =$$

$$= \left[\left(\operatorname{sen}\frac{3\pi}{2} - \cos\frac{3\pi}{2}\right) - \left(\operatorname{sen}\frac{\pi}{2} - \cos\frac{\pi}{2}\right)\right] \cdot \left(\frac{8}{3} - \frac{1}{3}\right)$$

$$= \left[(-1 - 0) - (1 - 0)\right] \cdot \frac{7}{3} = -\frac{14}{3}$$

L9

13. Calcule as integrais impróprias abaixo.

(a) $\displaystyle\int_0^\infty \int_0^\infty e^{-x-y}\,dx\,dy$

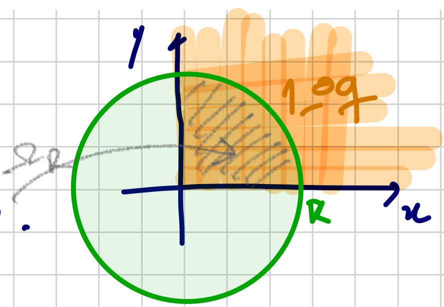
(b) $\displaystyle\int_0^\infty \int_0^\infty x^2 e^{-x^2-y^2}\,dx\,dy$

(c) $\displaystyle\iint_{x^2+y^2\leq 1} \frac{x^2\,dx\,dy}{(x^2+y^2)^{\frac{7}{4}}}$

(d) $\displaystyle\iint_{x^2+y^2\leq 1} \ln\sqrt{x^2+y^2}\,dx\,dy$

(b) Tome $R>0$ e considere

$$\Omega_R = \Omega \cap B_R(0).$$



Usaremos, então, coordenadas polares.

$$\int_0^\infty \int_0^\infty x^2 \cdot e^{-(x^2+y^2)}\,dx\,dy =$$

$$= \lim_{R\to\infty} \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{\rho=0}^{\rho=R} (\rho\cos\theta)^2 \cdot e^{-\rho^2} \cdot \rho\, d\rho\, d\theta =$$

$$= \lim_{R\to\infty} \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{\rho=0}^{\rho=R} \cos^2\theta \cdot e^{-\rho^2} \cdot \rho^3\, d\rho\, d\theta =$$

$$= \int_{\theta=0}^{\theta=\frac{\pi}{2}} \cos^2\theta\, d\theta \cdot \lim_{R\to\infty} \underbrace{\int_{\rho=0}^{\rho=R} \rho^3 \cdot e^{-\rho^2}\, d\rho}_{(*)} \quad =$$

$$(*): \int \rho^3 e^{-\rho^2} d\rho = -\frac{1}{2} \int \rho^2 e^{-\rho^2} \cdot (-2\rho\, d\rho) = \int u\, dv$$

$$\begin{cases} u = \rho^2 \Rightarrow du = 2\rho\, d\rho \\ dv = e^{-\rho^2} \cdot (-2\rho)\, d\rho \rightarrow v = e^{-\rho^2} \end{cases}$$

$$= -\frac{1}{2} \cdot \left[\rho^2 \cdot e^{-\rho^2} + \int e^{-\rho^2} (-2\rho\, d\rho) \right]$$

$$= -\frac{1}{2} \left[\rho^2 e^{-\rho^2} + e^{-\rho^2} + C \right]$$

$$= -\frac{1}{2} e^{-\rho^2} \cdot \left[\rho^2 + 1 \right] + C$$

$$= \int_{\theta=0}^{\theta=\frac{\pi}{2}} \underbrace{\cos^2\theta \, d\theta}_{\frac{1+\cos 2\theta}{2}} \lim_{R\to\infty} \left(-\frac{1}{2} e^{-\rho^2}\left[\rho^2+1\right]\right)\Bigg|_0^R =$$

L'HOPITAL.

$$= \int_{\theta=0}^{\theta=\frac{\pi}{2}} \frac{1+\cos 2\theta}{2} \, d\theta \cdot \lim_{R\to\infty}\left(-\underbrace{\cancel{\frac{R^2+1}{2e^{R^2}}}}_{\to 0} + \frac{1}{2e^0}\right)$$

$$= \left(\frac{1}{2}\int_0^{\frac{\pi}{2}} d\theta + \frac{1}{2}\cdot\frac{1}{2}\int_0^{\frac{\pi}{2}} \cos 2\theta \, (2\, d\theta)\right) \cdot \left(0 + \frac{1}{2}\right) =$$

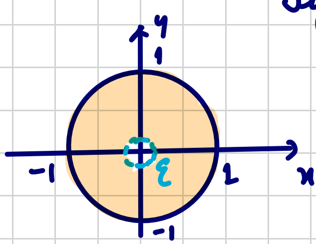
$$= \left(\frac{1}{2}\, \theta\Big|_0^{\frac{\pi}{2}} + \frac{1}{4}\cdot \operatorname{sen} 2\theta\Big|_0^{\frac{\pi}{2}}\right) \cdot \frac{1}{2} =$$

$$= \left(\frac{\pi}{4} - 0 + \frac{1}{4}\cdot\left(\underbrace{\operatorname{sen}\pi}_{=0} - \underbrace{\operatorname{sen} 0}_{=0}\right)\right)\cdot\frac{1}{2} = \frac{\pi}{4}\cdot\frac{1}{2} = \frac{\pi}{8}$$

(c) $\displaystyle\iint_{x^2+y^2\le 1} \frac{x^2\,dx\,dy}{(x^2+y^2)^{\frac{7}{4}}}$

Tome $0<\varepsilon<1$.

Fixe $\Omega_\varepsilon = B_1(0)\setminus B_\varepsilon(0)$.



Disso, teremos:

[VAMOS USAR COORD. POLARES]

$$\iint_{x^2+y^2\le 1} \frac{x^2\,dx\,dy}{(x^2+y^2)^{7/4}} = \lim_{\varepsilon\to 0^+} \int_{\theta=0}^{\theta=2\pi} \int_{\rho=\varepsilon}^{\rho=1} \frac{(\rho\cos\theta)^2 \rho \, d\rho \, d\theta}{(\rho^2)^{\cancel{7/4}\,7/2}}$$

$$\begin{cases} x = \rho\cos\theta \\ dx\,dy = \rho\,d\rho\,d\theta . \\ x^2+y^2 = \rho^2 \end{cases}$$

$$= \lim_{\varepsilon\to 0^+} \int_{\theta=0}^{\theta=2\pi} \int_{\rho=\varepsilon}^{\rho=1} \cos^2\theta \cdot \rho^3 \cdot \rho^{-\frac{7}{2}} \, d\rho\, d\theta =$$

$$= \int_{\theta=0}^{\theta=2\pi} \cos^2\theta \, d\theta \cdot \lim_{\varepsilon\to 0^+} \int_{\rho=\varepsilon}^{\rho=1} \rho^{-1/2} d\rho =$$

$$= \int_{\theta=0}^{\theta=2\pi} \frac{1+\cos 2\theta}{2} \, d\theta \cdot \lim_{\varepsilon\to 0^+} \left. \frac{\rho^{1/2}}{\frac{1}{2}} \right|_{\rho=\varepsilon}^{\rho=1} =$$

$$= \frac{1}{\cancel{2}} \int_{\theta=0}^{\theta=2\pi} (1+\cos 2\theta)\, d\theta \cdot \cancel{2} \cdot \lim_{\varepsilon\to 0^+} \left. \sqrt{\rho} \right|_{\rho=\varepsilon}^{\rho=1} =$$

$$= \left(\int_{\theta=0}^{\theta=2\pi} d\theta + \frac{1}{2} \cdot \int_{\theta=0}^{\theta=2\pi} \cos 2\theta \, 2\, d\theta \right) \cdot \lim_{\varepsilon\to 0^+} \left(\sqrt{1} - \underbrace{\sqrt{\varepsilon}}_{\to 0} \right) =$$

$$\left(\theta\Big|_0^{2\pi} + \frac{1}{2}\,\operatorname{sen}2\theta\Big|_0^{2\pi}\right)\cdot 1 = 2\pi + \frac{1}{2}\left(\operatorname{sen}4\pi - \operatorname{sen}0\right)$$

$$= 2\pi$$

LG

6. Calcule $\displaystyle\iint_{\Omega} \arctan\frac{y}{x}\,dA$, onde Ω é a região do primeiro quadrante compreendida entre dois círculos $x^2+y^2=4$ e $x^2+y^2=2x$.

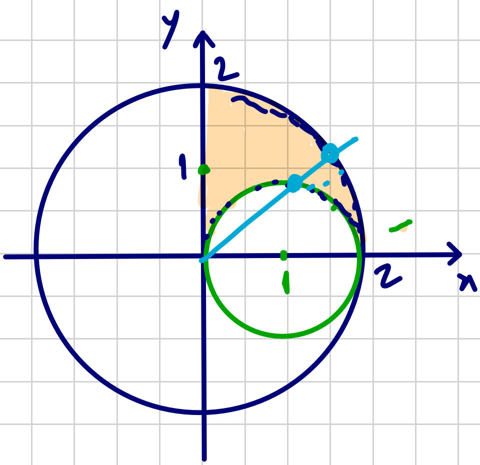
$$x^2-2x+y^2=0$$

$$\underbrace{x^2-2x+1}+y^2=1$$

$$(x-1)^2+y^2=1.$$

centro: $(1,0)$

COORD. POLARES



$$0 \leq \theta \leq \frac{\pi}{2}$$

PONTOS DA CIRCUNF. MENOR $\leq \rho \leq 2$

$$x^2+y^2=2x$$

$$\rho^{\cancel{2}} = 2\cancel{\rho}\cos\theta \Rightarrow \rho = 2\cos\theta$$

$\Rightarrow 2\cos\theta \leq \rho \leq 2$

Dessa forma, vamos obter:

$\tan\theta = \frac{y}{x}$

$\theta = \arctan\frac{y}{x}$

$$\iint_{\Omega} \arctan\frac{y}{x}\, dx\, dy =$$

$$= \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{\rho=2\cos\theta}^{\rho=2} \theta \cdot \rho\, d\rho\, d\theta =$$

$$= \int_{\theta=0}^{\theta=\frac{\pi}{2}} \theta \cdot \left(\int_{\rho=2\cos\theta}^{\rho=2} \rho\, d\rho \right) d\theta = \int_{\theta=0}^{\theta=\frac{\pi}{2}} \theta \cdot \frac{\rho^2}{2} \Big|_{\rho=2\cos\theta}^{\rho=2} d\theta =$$

$$= \int_{\theta=0}^{\theta=\frac{\pi}{2}} \theta \cdot \left(\frac{4}{2} - \frac{4\cos^2\theta}{2} \right) d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \theta\, d\theta - 2\int_0^{\frac{\pi}{2}} \theta \cos^2\theta\, d\theta =$$

$$= 2\,\theta \Big|_0^{\frac{\pi}{2}} - 2\underbrace{\int_0^{\frac{\pi}{2}} \cos^2\theta \cdot \theta\, d\theta}_{\to \text{ integrar por partes.}}$$

$$\int \cos^2\theta \cdot \theta \, d\theta = \int u \, dv = u \cdot v - \int v \, du \ominus$$

$$u = \theta \Rightarrow du = d\theta.$$

$$dv = \cos^2\theta \, d\theta \Rightarrow v = \int \cos^2\theta \, d\theta = \int \frac{1+\cos 2\theta}{2} d\theta$$

$$= \frac{1}{2}\int d\theta + \frac{1}{2}\frac{1}{2}\int \cos 2\theta \, 2 \, d\theta$$

$$= \frac{\theta}{2} + \frac{1}{4} \cdot \operatorname{sen} 2\theta$$

$$\ominus \ \theta \cdot \left(\frac{\theta}{2} + \frac{1}{4}\operatorname{sen}2\theta\right) - \int \left(\frac{\theta}{2} + \frac{1}{4}\operatorname{sen}2\theta\right) d\theta$$

$$= \frac{\theta^2}{2} + \frac{\theta}{4} \cdot \operatorname{sen}2\theta - \frac{1}{2}\int \theta \, d\theta - \frac{1}{4}\int \operatorname{sen}2\theta \, d\theta$$

$$= \cancel{\frac{\theta^2}{2}} + \frac{\theta}{4}\operatorname{sen}2\theta - \cancel{\frac{\theta^2}{2}} + \frac{1}{8}\cos 2\theta + c$$

$$= \frac{\theta}{4} \cdot \operatorname{sen}2\theta + \frac{1}{8}\cos 2\theta + c$$

$$\ominus \left(\frac{\pi}{2}\right)^2 - 2 \cdot \left(\frac{\theta}{4}\operatorname{sen}2\theta + \frac{1}{8}\cos 2\theta\right)\Bigg|_0^{\frac{\pi}{2}} =$$

$$= \frac{\pi^2}{4} - 2 \cdot \left(\frac{\pi}{8}\underbrace{(\operatorname{sen}\pi - \operatorname{sen}0)}_{=0} + \frac{1}{8}(\underbrace{\cos\pi}_{-1} - \cos 0)\right)$$

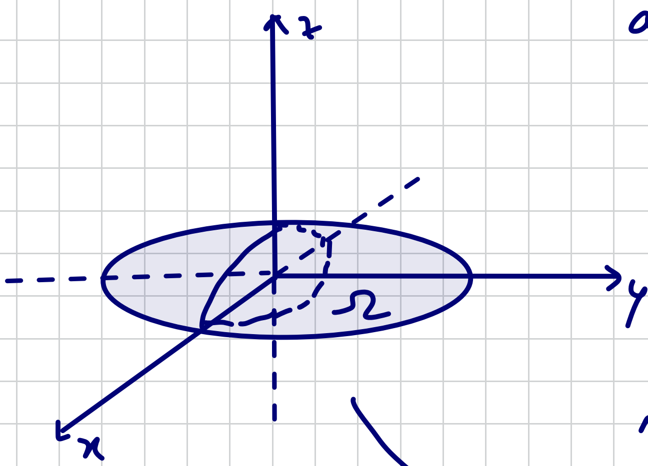
$$= \frac{\pi^2}{4} - 2 \cdot \left(0 + \frac{1}{8} \cdot (-2) \right) =$$

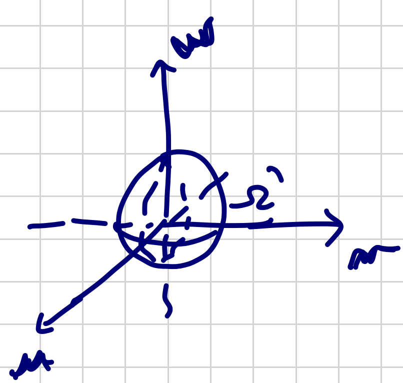
$$= \frac{\pi^2}{4} + \frac{1}{2}$$

LISTA 10:

14. Determine o volume da região interior ao elipsóide $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1.$ (Resp.: $\frac{4\pi}{3}$ u.v.)

$a, b, c > 0.$





Vamos passar do sistema xyz para o sistema uvw transformando o elipsóide numa esfera.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2=1.$$

$$\left.\begin{aligned} u &= \frac{x}{a} \\ v &= \frac{y}{b} \\ w &= \frac{z}{c} \end{aligned}\right\} \quad u^2+v^2+w^2=1.$$

(esfera de raio unitário centrada na origem.

Assim:

$$V=\iiint_{\Omega} dV=\iiint_{\Omega} 1.\,dx\,dy\,dz=$$

$$=\iiint_{\Omega'} 1.\,|\det J(T)(u,v,w)|\,du\,dv\,dw$$

$$J(T)(u,v,w)=\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix}$$

Como $u=\frac{x}{a}$, $v=\frac{y}{b}$ e $w=\frac{z}{c}$, temos: $\begin{cases} x = a.u \\ y = b.v \\ z = c.w \end{cases}$. Então:

$$J(T)(u,v,w) = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

$$\Rightarrow \det J(T)(u,v,w) = a.b.c$$

$$\Rightarrow |\det J(T)(u,v,w)| = |a.bc| = a.b.c.$$

Assim, teremos:

$$V = \iiint_{\Omega'} \underbrace{|\det J(T)(u,v,w)|}_{a.bc}.du\,dv\,dw =$$

$$= a.bc \underbrace{\iiint_{\Omega'} du\,dv\,dw}_{\text{VOL. DA ESFERA DE RAIO 1.}} = a.bc.\frac{4\pi}{3} = \frac{4\pi}{3}abc$$

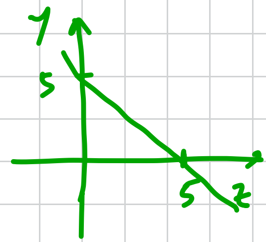
COORD. ESFÉRICAS

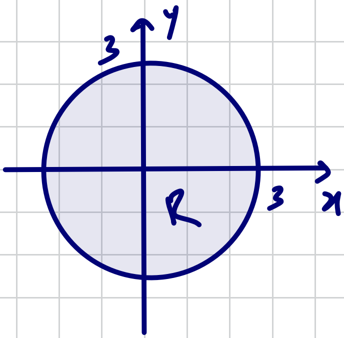
LISTA 10:

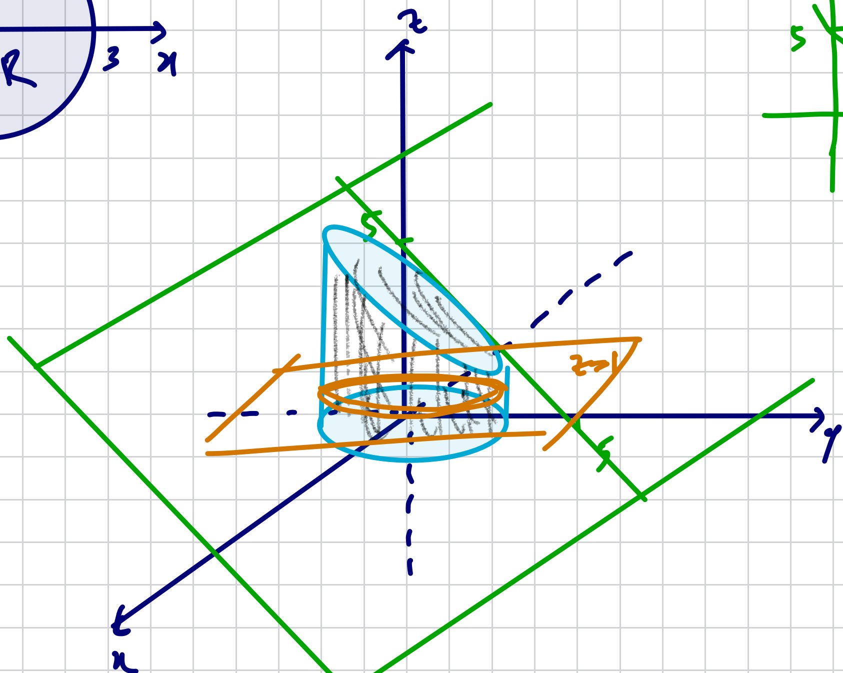
7. Use integral tripla para determinar o volume do sólido dado em cada item.

   (a) O sólido limitado pelo cilindro $x^2 + y^2 = 9$ e pelos planos $y + z = 5$ e $z = 1$. (Resp.: 36π u.v.)

$z = 5 - y$







$$V = \iiint_{\Omega} dV = \iint_{R} dx\,dy \int_{z=1}^{z=5-y} dz =$$

COORD. CILÍNDRICAS

$$= \int_{\theta=0}^{\theta=2\pi} \int_{\rho=0}^{\rho=3} \rho \underbrace{\left(\int_{z=1}^{z=5-\rho \operatorname{sen}\theta} dz \right)}_{f(\rho,\theta)} d\rho \, d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{\rho=0}^{\rho=3} \rho \cdot z \Big|_{z=1}^{z=5-\rho \operatorname{sen}\theta} d\rho \, d\theta = \int_{\theta=0}^{\theta=2\pi} \int_{\rho=0}^{\rho=3} \rho \cdot (5-\rho \operatorname{sen}\theta - 1) \, d\rho \, d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{\rho=0}^{\rho=3} (4\rho - \rho^2 \operatorname{sen}\theta) \, d\rho \, d\theta = \int_{\theta=0}^{\theta=2\pi} \left(\frac{4\rho^2}{2} - \frac{\rho^3}{3} \operatorname{sen}\theta \right) \Big|_{\rho=0}^{\rho=3} d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \left(2\rho^2 - \frac{\rho^3}{3} \operatorname{sen}\theta \right) \Big|_{\rho=0}^{\rho=3} d\theta = \int_0^{2\pi} (18 - 9 \operatorname{sen}\theta) \, d\theta =$$

$$(18\theta + 9\cos\theta) \Big|_0^{2\pi} = 18 \cdot (2\pi - 0) + \underbrace{9 \cdot (\cos 2\pi - \cos 0)}_{=0}$$

$$= 36\pi \ \underline{\underline{u.v.}}$$