CÁLCULO 3, TURMA T3

17/07/26 – AULA 45

AULA DE EXERCÍCIOS:

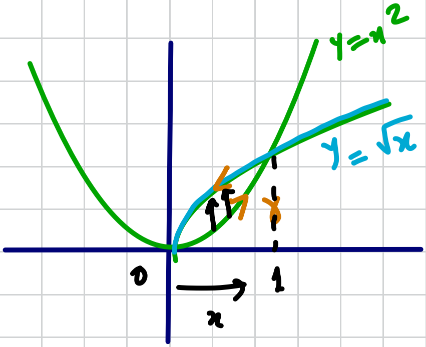
LISTA 12

8. Use o teorema de Green para calcular cada integral de linha a seguir, ao longo da curva dada com orientação positiva.

(a) $\int_\gamma e^y dx + 2xe^y dy$, onde γ é o quadrado de lados $x = 0$, $x = 1$, $y = 0$ e $y = 1$.

(b) $\int_\gamma (ye^{\sqrt{x}})dx + (2x + \cos y^2)dy$, onde γ é a fronteira da região delimitada pelas parábolas $y = x^2$ e $x = y^2 \rightarrow y = \sqrt{x}$

(b) $$\oint_\gamma \underbrace{(y \cdot e^{\sqrt{x}})}_{P} dx + \underbrace{(2x + \cos y^2)}_{Q} dy$$



$$\begin{cases} P = ye^{\sqrt{x}} \Rightarrow \dfrac{\partial P}{\partial y} = e^{\sqrt{x}} \\ Q = 2x + \cos y^2 \Rightarrow \dfrac{\partial Q}{\partial x} = 2 \end{cases}$$

Pelo T. de Green:

$$\oint_\gamma P\,dx + Q\,dy = \iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \int_{x=0}^{x=1} \int_{y=x^2}^{y=\sqrt{x}} \underbrace{\left(2 - e^{\sqrt{x}}\right)}_{\text{CONST. PARA } y} dy\, dx =$$

$$= \int_{x=0}^{x=1} \left(2 - e^{\sqrt{x}}\right)\left(\int_{y=x^2}^{y=\sqrt{x}} dy\right) dx =$$

$$= \int_{x=0}^{x=1} \left(2 - e^{\sqrt{x}}\right) \cdot \left(\sqrt{x} - x^2\right) dx =$$

$$= \int_0^1 \left(2\sqrt{x} - 2x^2 - \sqrt{x} \cdot e^{\sqrt{x}} + x^2 e^{\sqrt{x}}\right) dx$$

$$= \int_0^1 2x^{\frac{1}{2}} dx - \int_0^1 2x^2 dx - \underbrace{\int_0^1 \sqrt{x}\, e^{\sqrt{x}} dx}_{(*)} + \underbrace{\int_0^1 x^2 e^{\sqrt{x}} dx}_{(**)}$$

MAIS DIFÍCEIS

$(*)$: $\int \sqrt{x} \cdot e^{\sqrt{x}} dx = \int w \cdot e^{\sqrt{w^2}} \cdot 2w\, dw =$

$\sqrt{x} = w \Rightarrow x = w^2$

$dx = 2w\, dw$

$$= 2 \cdot \int w^2 \cdot e^w dw = 2\left(\int u\, dv\right) = 2\left(u \cdot v - \int v\, du\right) = $$

$u = w^2 \Rightarrow du = 2w\, dw$

$$dv = e^{w} \cdot dw \Rightarrow v = e^{w}$$

$$= 2 \cdot \left(w^2 e^{w} - \int e^{w} \cdot 2w\, dw \right) =$$

$$= 2w^2 e^{w} - 4 \underbrace{\int w \cdot e^{w}\, dw}_{\int u\,dv = u \cdot v - \int v\,du} =$$

$$\begin{cases} u = w \Rightarrow du = dw \\ dv = e^{w} dw \Rightarrow v = e^{w} \end{cases}$$

$$= 2w^2 e^{w} - 4 \cdot \left(w \cdot e^{w} - \int e^{w} \cdot dw \right) =$$

$$= 2w^2 e^{w} - 4w e^{w} + 4 \cdot e^{w} + c$$

$$= 2e^{w} \left(w^2 - 2w + 2 \right) + c =$$

$w = \sqrt{x}$

$$= 2e^{\sqrt{x}} \left(x - 2\sqrt{x} + 2 \right) + c$$

$$\Rightarrow \int_0^1 \sqrt{x}\, e^{\sqrt{x}}\, dx = 2e\left(1 \cancel{-2} \cancel{+2}\right) - 2e^0(0 - 0 + 2)$$

$$= \underline{2 \cdot e - 4}$$

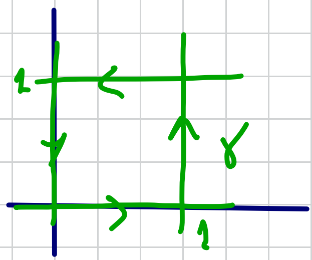
(**) $\displaystyle\int x^2 \cdot e^{\sqrt{x}}\,dx \underset{\substack{\uparrow \\ w^2 = x \;\Rightarrow\; dx = 2w\,dw}}{=} \int w^4 \cdot e^{w}\,(2w\,dw) =$

$$= \underbrace{2\int w^5 e^{w}\,dw}_{} =$$

VAMOS PRECISAR INTEGRAR POR PARTES CINCO VEZES

Então, deixemos de lado...

(a) $\displaystyle\oint_\gamma \underbrace{e^y}_{P}\,dx + \underbrace{2x e^y}_{Q}\,dy$



$$\oint_\gamma P\,dx + Q\,dy = \iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_\Omega \left(2e^y - e^y\right) dA =$$

$$\begin{cases} P = e^y \Rightarrow \dfrac{\partial P}{\partial y} = e^y \\[2ex] Q = 2x e^y \Rightarrow \dfrac{\partial Q}{\partial x} = 2 \cdot e^y \end{cases}$$

$$= \iint_{\Omega} e^{y} \cdot dy\, dx = \int_{x=0}^{x=1}\int_{y=0}^{y=1} e^{y}\, dy\, dx =$$

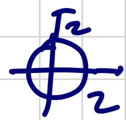
$$= \int_{x=0}^{x=1} dx \cdot \int_{y=0}^{y=1} e^{y} dy = x\Big|_0^1 \cdot e^{y}\Big|_0^1 =$$

$$= \underbrace{(1-0)}_{1} \cdot (e^{1} - e^{0}) = e - 1$$

L12:

10. Use o Teorema de Green para calcular $\oint_\gamma \underbrace{e^{x+y}}_{P} dx + \underbrace{e^{x+y}}_{Q} dy$, onde γ é a circunferência $x^2 + y^2 = 4$.

$$\begin{cases} P = e^{x+y} \Rightarrow \dfrac{\partial P}{\partial y} = e^{x+y} \cdot 1 = e^{x+y} \\[2ex] Q = e^{x+y} \Rightarrow \dfrac{\partial Q}{\partial x} = e^{x+y} \cdot 1 = e^{x+y} \end{cases}$$

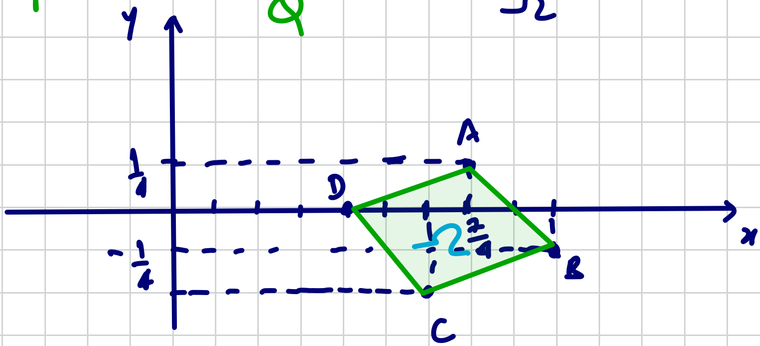


$$\Rightarrow \oint_\gamma P\,dx + Q\,dy = \iint_{\Omega} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA =$$

$$= \iint_{\Omega} 0\, dA = 0$$

9. Utilize o Teorema de Green para calcular $\oint_\gamma \cos(x-3y)dx + \ln(x+y)dy$, onde γ é o quadrilátero $ABCD$ de vértices $A(\frac{7}{4},\frac{1}{4})$, $B(\frac{9}{4},-\frac{1}{4})$, $C(\frac{3}{2},-\frac{1}{2})$ e $D(1,0)$.

$$\oint_\gamma \underbrace{\cos(x-3y)}_{P} + \underbrace{\ln(x+y)}_{Q}\,dy = \iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \; =$$



$$P = \cos(x-3y) \Rightarrow \underbrace{\frac{\partial P}{\partial y}} = -\text{sen}(x-3y)\cdot(-3) = \underbrace{3\,\text{sen}(x-3y)}$$

$$Q = \ln(x+y) \Rightarrow \underbrace{\frac{\partial Q}{\partial x} = \frac{1}{x+y}}$$

$$= \iint_\Omega \left(\frac{1}{\underbrace{x+y}_{u}} - 3\,\text{sen}(\underbrace{x-3y}_{v})\right) dA \; =$$

Vamos precisar, agora, mudar de variável. Escreva:

$$-\begin{cases} u = x+y \\ v = x-3y \end{cases}$$

$$u - v = \cancel{x} + y - \cancel{x} + 3y \Rightarrow \boxed{y = \frac{1}{4}u - \frac{1}{4}v}$$

$$u = x + y \Rightarrow x = u - y$$

$$x = u - \left(\frac{1}{4}u - \frac{1}{4}v\right)$$

$$\boxed{x = \frac{3}{4}u + \frac{1}{4}v}$$

$$= \iint_{\Omega'} \left(\frac{1}{u} - 3.\operatorname{sen} v\right).\left|\det J(T)(u,v)\right|.du\,dv \; \boxed{=}$$

$$J(T) = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

$$\Rightarrow \det J(T)(u,v) = \begin{vmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} \end{vmatrix} =$$

$$= \frac{3}{4}.\left(-\frac{1}{4}\right) - \left(\frac{1}{4}\right).\left(\frac{1}{4}\right) =$$

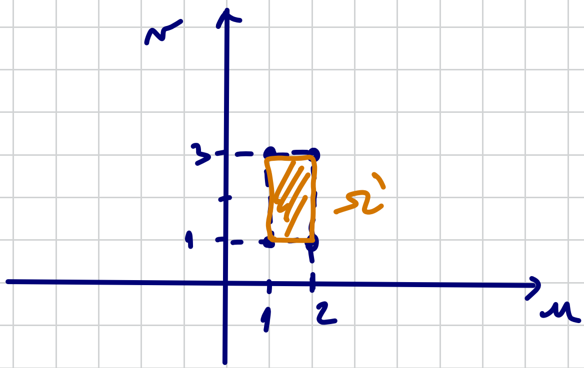
$$= -\frac{3}{16} - \frac{1}{16} = -\frac{4}{16} = -\frac{1}{4}$$

Assim, temos:

$$\boxed{=} \iint_{\Omega'} \left(\frac{1}{u} - 3.\operatorname{sen} v\right).\left|-\frac{1}{4}\right|.du\,dv =$$

$$\frac{1}{4}\iint_{\Omega'} \left(\frac{1}{u} - 3\operatorname{sen} v\right)du\,dv \; \triangle\!\!=$$

(x,y)	$(u,v)=(x+y,\ x-3y)$
$A\left(\frac{7}{4},\frac{1}{4}\right)$	$T(A)=\left(\frac{7}{4}+\frac{1}{4},\ \frac{7}{4}-3\cdot\frac{1}{4}\right)=(2,1)$
$B\left(\frac{9}{4},-\frac{1}{4}\right)$	$T(B)=\left(\frac{9}{4}-\frac{1}{4},\ \frac{9}{4}+\frac{3}{4}\right)=(2,3)$
$C\left(\frac{3}{2},-\frac{1}{2}\right)$	$T(C)=\left(\frac{3}{2}-\frac{1}{2},\ \frac{3}{2}+\frac{3}{2}\right)=(1,3)$
$D(1,0)$	$T(D)=(1+0,\ 1-3\cdot 0)=(1,1)$



$$= \frac{1}{4}\int_{v=1}^{v=3}\int_{u=1}^{u=2}\left(\frac{1}{u}-3\,\mathrm{sen}\,v\right)du\,dv =$$

$$= \frac{1}{4}\int_{v=1}^{v=3}\left(\ln|u| - 3\,\mathrm{sen}\,v\cdot u\right)\Big|_{u=1}^{u=2} dv$$

$$= \frac{1}{4}\int_{v=1}^{v=3}\left[\left(\ln 2 - \underbrace{\ln 1}_{=0}\right) - 3\,\mathrm{sen}\,v\cdot\underbrace{(2-1)}_{=1}\right]dv$$

$$= \frac{1}{4} \cdot \int_1^3 (\ln 2 - 3 \cdot \operatorname{sen} v)\, dv =$$

$$= \frac{1}{4} \cdot \ln 2 \cdot \int_1^3 dv - \frac{3}{4} \int_1^3 \operatorname{sen} v \, dv$$

$$= \frac{1}{4} \ln 2 \cdot v \Big|_1^3 - \frac{3}{4} \cdot (-\cos v) \Big|_1^3 =$$

$$= \frac{1}{4} \ln 2 \cdot (3-1) - \frac{3}{4} \cdot (-\cos 3 - (-\cos 1))$$

$$= \frac{1}{2} \ln 2 - \frac{3}{4} \cdot (\cos 1 - \cos 3)$$