CÁLCULO 3, TURMA T3

01/07/26 – AULA 38

No final da aula passada estudamos o sistema de coordenadas cilíndricas.

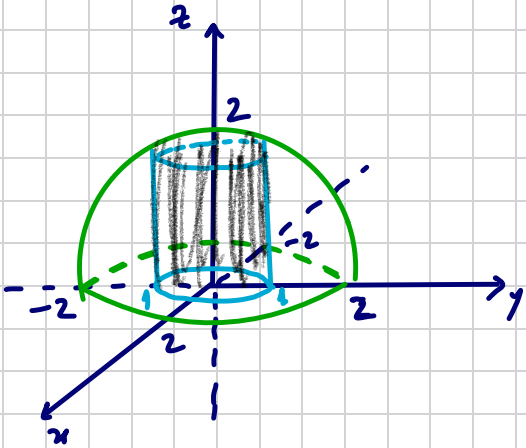
$$P(x,y,z) \leftrightarrow P(\rho,\theta,z);$$

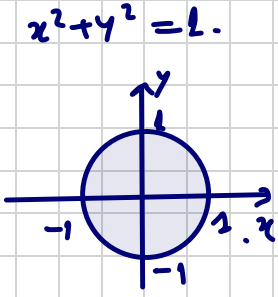
onde $\begin{cases} x = \rho\cos\theta \\ y = \rho\operatorname{sen}\theta \\ z = z \end{cases}$

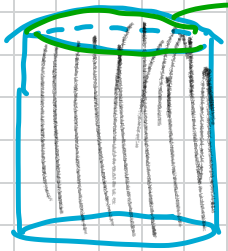
Vejamos um exemplo de aplicação.

Ex.: Calcule o volume do sólido abaixo da semi-esfera $x^2+y^2+z^2=4$, $z \geq 0$, limitado pelo cilindro $x^2+y^2=1$ e pelo plano xy, usando integrais triplas.

SOLUÇÃO:







$x^2+y^2+z^2=4$

$z=\sqrt{4-x^2-y^2}$

$V=?$

$\sqrt{4-\rho^2}$

$x^2+y^2=\rho^2$

$$V=\iiint_{\Omega} dV=\iint_{R} dx\,dy\int_{z=0}^{z=\sqrt{4-x^2-y^2}} dz =$$

circular.

$$=\int_{\theta=0}^{\theta=2\pi}\int_{\rho=0}^{\rho=1}\rho\,d\rho\,d\theta\cdot\int_{z=0}^{z=\sqrt{4-\rho^2}} dz = \int_{\theta=0}^{\theta=2\pi}\int_{\rho=0}^{\rho=1} z\Big|_0^{\sqrt{4-\rho^2}}\rho\,d\rho\,d\theta$$

$$=\int_{\theta=0}^{\theta=2\pi}\int_{\rho=0}^{\rho=1}\sqrt{4-\rho^2}\cdot\rho\,d\rho\,d\theta = \int_{\theta=0}^{\theta=\pi} d\theta\left(-\frac{1}{2}\right)\int_{\rho=0}^{\rho=1}(4-\rho^2)^{\frac{1}{2}}(-2\rho\,d\rho)$$

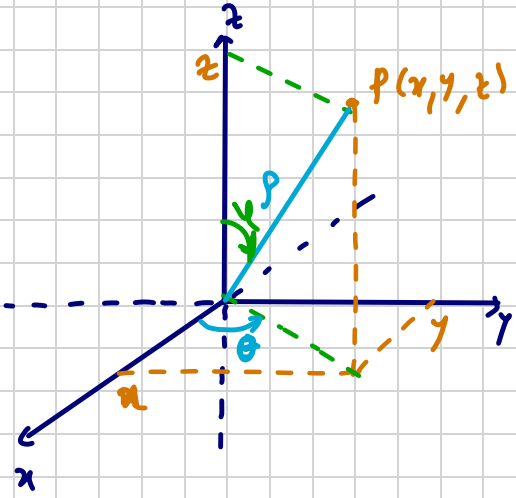
$r=4-\rho^2\Rightarrow dr=-2\rho\,d\rho$

$$=-\frac{1}{2}\cdot\theta\Big|_0^{2\pi}\cdot\frac{(4-\rho^2)^{\frac{3}{2}}}{\frac{3}{2}}\Big|_0^1 = -\frac{1}{2}\cdot 2\pi\cdot\frac{2}{3}\left[(3)^{\frac{3}{2}}-(4)^{\frac{3}{2}}\right]$$

$$= -\frac{2}{3}\pi\left((3)^{\frac{3}{2}} - (4)^{\frac{3}{2}}\right) = \frac{2\pi}{3}\left(4^{\frac{3}{2}} - 3^{\frac{3}{2}}\right) u.m.$$

SISTEMA DE COORDENADAS ESFÉRICAS

A ideia consiste em pensar em pontos localizados sobre a superfície de uma esfera.



$P(x,y,z)$

$\updownarrow$

$P(\rho, \varphi, \theta)$,

onde:

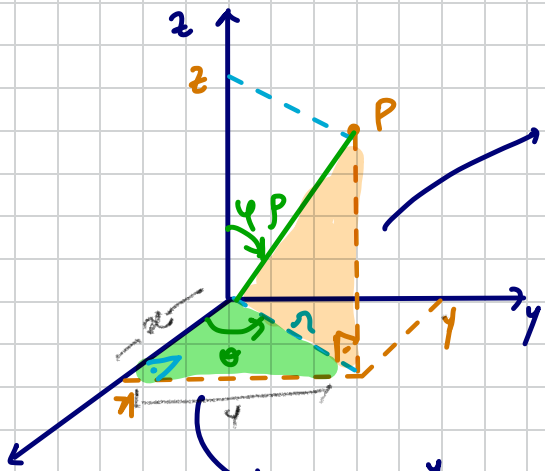
$\rho \in \mathbb{R}$ (raio da "esfera")

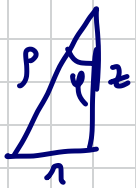
$0 < \theta \leq 2\pi$

$0 < \varphi \leq \pi$

↳ φ é o ângulo partindo do semi-eixo positivo z até o semi-eixo negativo z.

Relações de conversão do sistema retangular xyz para o esférico $\rho\varphi\theta$:



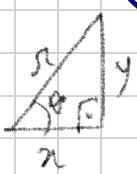


$\operatorname{sen}\varphi = \frac{r}{\rho}$

$\boxed{r = \rho \operatorname{sen}\varphi}$

$\cos\varphi = \frac{z}{\rho}$

$\Rightarrow \boxed{z = \rho\cos\varphi}$



$\operatorname{sen}\theta = \frac{y}{r} \Rightarrow y = r \cdot \operatorname{sen}\theta$

$\cos\theta = \frac{x}{r} \Rightarrow x = r \cdot \cos\theta$

Assim, obtemos:

$$x = r\cos\theta = \rho\operatorname{sen}\varphi \cdot \cos\theta$$

$$y = r\operatorname{sen}\theta = \rho \cdot \operatorname{sen}\varphi \cdot \operatorname{sen}\theta$$

$$z = \rho\cos\varphi$$

Ou seja, a transformação T do sistema xyz para o sistema $\rho\varphi\theta$ é dada por:

$$T: \begin{cases} x = \rho \operatorname{sen}\varphi \cos\theta \\ y = \rho \operatorname{sen}\varphi \operatorname{sen}\theta \\ z = \rho \cos\varphi \end{cases}$$

Neste caso, uma esfera de raio $R>0$, centrada na origem de $\mathbb{R}^3$, no sistema esférico fica representada por:

$$x^2+y^2+z^2 = R^2$$

$$(\rho \operatorname{sen}\varphi \cos\theta)^2 + (\rho \operatorname{sen}\varphi \operatorname{sen}\theta)^2 + (\rho\cos\varphi)^2 = R^2$$

$$\rho^2 \operatorname{sen}^2\varphi \cos^2\theta + \rho^2 \operatorname{sen}^2\varphi \operatorname{sen}^2\theta + \rho^2\cos^2\varphi = R^2$$

$$\rho^2 \operatorname{sen}^2\varphi (\underbrace{\cos^2\theta + \operatorname{sen}^2\theta}_{=1}) + \rho^2\cos^2\varphi = R^2$$

$$\rho^2 \operatorname{sen}^2\varphi + \rho^2\cos^2\varphi = R^2$$

$$\rho^2 (\underbrace{\operatorname{sen}^2\varphi + \cos^2\varphi}_{=1}) = R^2 \Rightarrow \rho^2 = R^2 \Rightarrow \boxed{\rho = R}$$

Seja $f: \Omega \subset \mathbb{R}^3 \to \mathbb{R}$ uma função integrável, Ω é delimitada por esfera. Então:

$$\iiint_{\Omega} f(x,y,z)\,dV =$$

$$= \iiint_{\Omega'} f(x(\rho,\varphi,\theta), y(\rho,\varphi,\theta), z(\rho,\varphi,\theta)) \cdot |\det J(T)(\rho,\varphi,\theta)|\, d\rho\, d\varphi\, d\theta$$

Vamos determinar a expressão para $|\det J(T)(\rho,\varphi,\theta)|$.

Como

$$T: \begin{cases} x = \rho \operatorname{sen}\varphi \cos\theta \\ y = \rho \operatorname{sen}\varphi \operatorname{sen}\theta \\ z = \rho \cos\varphi \end{cases},$$ então:

$$J(T)(\rho,\varphi,\theta) = \begin{bmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial y}{\partial \rho} & \frac{\partial z}{\partial \rho} \\ \frac{\partial x}{\partial \varphi} & \frac{\partial y}{\partial \varphi} & \frac{\partial z}{\partial \varphi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \end{bmatrix} =$$

$$= \begin{bmatrix} \operatorname{sen}\varphi\cos\theta & \operatorname{sen}\varphi\operatorname{sen}\theta & \cos\varphi \\ \rho\cos\varphi\cos\theta & \rho\cos\varphi\operatorname{sen}\theta & -\rho\operatorname{sen}\varphi \\ -\rho\operatorname{sen}\varphi\operatorname{sen}\theta & \rho\operatorname{sen}\varphi\cos\theta & 0 \end{bmatrix}$$

Assim, teremos:

$$\det J(T)(\rho,\varphi,\theta) =$$

$$\begin{vmatrix} \operatorname{sen}\varphi\cos\theta & \operatorname{sen}\varphi\operatorname{sen}\theta & \cos\varphi \\ \rho\cos\varphi\cos\theta & \rho\cos\varphi\operatorname{sen}\theta & -\rho\operatorname{sen}\varphi \\ -\rho\operatorname{sen}\varphi\operatorname{sen}\theta & \rho\operatorname{sen}\varphi\cos\theta & 0 \end{vmatrix} \begin{matrix} \operatorname{sen}\varphi\cos\theta & \operatorname{sen}\varphi\operatorname{sen}\theta \\ \rho\cos\varphi\cos\theta & \rho\cos\varphi\operatorname{sen}\theta \\ -\rho\operatorname{sen}\varphi\operatorname{sen}\theta & \rho\operatorname{sen}\varphi\cos\theta \end{matrix}$$

$$= 0 + \rho^2\operatorname{sen}^3\varphi\operatorname{sen}^2\theta + \rho^2\operatorname{sen}\varphi\cos^2\varphi\cos^2\theta +$$

$$+ \rho^2\operatorname{sen}\varphi\cos^2\varphi\cdot\operatorname{sen}^2\theta + \rho^2\cdot\operatorname{sen}^3\varphi\cos^2\theta - 0$$

$$= \rho^2\operatorname{sen}\varphi\left[\operatorname{sen}^2\varphi\cdot\operatorname{sen}^2\theta + \cos^2\varphi\cdot\cos^2\theta + \cos^2\varphi\cdot\operatorname{sen}^2\theta + \operatorname{sen}^2\varphi\cos^2\theta\right]$$

$$= \rho^2\operatorname{sen}\varphi\left[\operatorname{sen}^2\varphi\left(\operatorname{sen}^2\theta+\cos^2\theta\right) + \cos^2\varphi\left(\cos^2\theta+\operatorname{sen}^2\theta\right)\right]$$

$$= \rho^2\operatorname{sen}\varphi\left[\operatorname{sen}^2\varphi + \cos^2\varphi\right] = \rho^2\operatorname{sen}\varphi$$

$$\Rightarrow \det J(T)(\rho,\varphi,\theta) = \rho^2\operatorname{sen}\varphi$$

$$\Rightarrow \left|\det J(T)(\rho,\varphi,\theta)\right| = \left|\rho^2\operatorname{sen}\varphi\right| = \rho^2\cdot\operatorname{sen}\varphi$$

$0 \le \varphi \le \pi$

conclusão:

$$\iiint_{\Omega} f(x,y,z)\,dV =$$

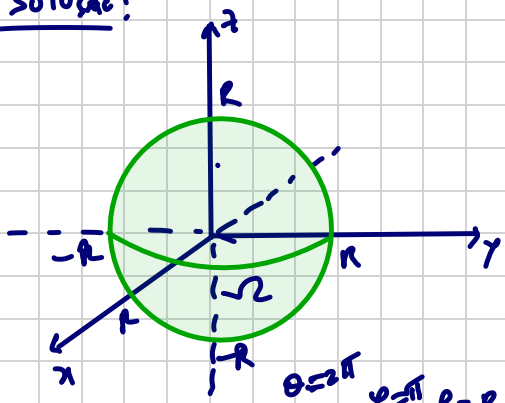
$$= \iiint_{\Omega'} f(x(\rho,\varphi,\theta), y(\rho,\varphi,\theta), z(\rho,\varphi,\theta)) \cdot \rho^2 \operatorname{sen}\varphi \, d\rho\, d\varphi\, d\theta$$

FÓRMULA DA MUDANÇA NO SISTEMA ESFÉRICO.

EXEMPLOS:

01) Use o sistema de coordenadas esféricas para deduzir a fórmula do volume de uma esfera de raio $R>0$.

SOLUÇÃO:



$$V = \iiint_{\Omega} dV = \int_{\theta=0}^{\theta=2\pi} \int_{\varphi=0}^{\varphi=\pi} \int_{\rho=0}^{\rho=R} \rho^2 \operatorname{sen}\varphi \, d\rho\, d\varphi\, d\theta =$$

$$= \int_{\theta=0}^{\theta=2\pi} d\theta \cdot \int_{\varphi=0}^{\varphi=\pi} \operatorname{sen}\varphi \, d\varphi \cdot \int_{\rho=0}^{\rho=R} \rho^2 \, d\rho$$

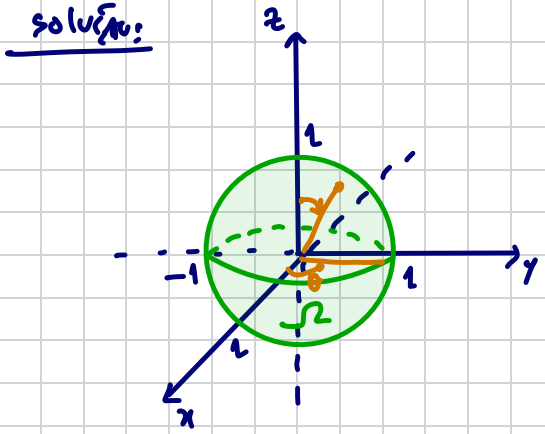
$$= \theta\Big|_{\theta=0}^{\theta=2\pi} \cdot (-\cos\varphi)\Big|_{\varphi=0}^{\varphi=\pi} \cdot \left(\frac{\rho^3}{3}\right)\Big|_{\rho=0}^{\rho=R} =$$

$$= 2\pi \cdot \underbrace{(-\cos\pi}_{+1} + \underbrace{\cos 0)}_{1} \cdot \frac{R^3}{3} = 4\pi \cdot \frac{R^3}{3} = \frac{4\pi R^3}{3}$$

02) Calcule $\iiint_{\Omega} e^{(x^2+y^2+z^2)^{\frac{3}{2}}} dV$, sendo Ω a região dada por:

$$\Omega = \{(x,y,z) \in \mathbb{R}^3 : \; x^2+y^2+z^2 \leq 1\}.$$

Solução:



$x^2+y^2+z^2 = \rho^2$

$0 \leq \rho \leq 1.$

$0 \leq \theta \leq 2\pi \qquad 0 \leq \varphi \leq \pi$

$$\iiint_{\Omega} e^{(x^2+y^2+z^2)^{3/2}} \cdot dV = \iiint_{\Omega'} e^{(\rho^2)^{3/2}} \cdot \rho^2 \operatorname{sen}\varphi \, \rho \, d\rho \, d\varphi \, d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{\varphi=0}^{\varphi=\pi} \int_{\rho=0}^{\rho=1} e^{\rho^3} \cdot \rho^2 \cdot \operatorname{sen}\varphi \cdot d\rho \, d\varphi \, d\theta =$$

$$= \int_{\theta=0}^{\theta=2\pi} d\theta \int_{\varphi=0}^{\varphi=\pi} \operatorname{sen}\varphi \, d\varphi \cdot \frac{1}{3} \int_{\rho=0}^{\rho=1} e^{\rho^3} (3\rho^2 d\rho) =$$

$$\int e^{u} du \qquad u = \rho^3 \qquad du = 3\rho^2 d\rho$$

$$= \theta \Big|_0^{2\pi} \cdot (-\cos\varphi) \Big|_0^{\pi} \cdot \frac{1}{3} \cdot e^{\rho^3} \Big|_0^1 =$$

$$= 2\pi \cdot \underbrace{(-\cos\pi + \cos 0)}_{2} \cdot \frac{1}{3} \cdot (e^1 - \underbrace{e^0}_{=1}) = \frac{4\pi}{3} \cdot (e-1)$$

03) Use coordenadas esféricas para determinar o volume do sólido acima do cone $z = \sqrt{x^2+y^2}$ e abaixo da esfera $x^2+y^2+z^2 = z$.

<u>solução:</u> Note que a esfera não está centrada na origem.

$$x^2+y^2+z^2 = z$$

$$x^2+y^2+z^2-z = 0$$

→ PRECISAMOS CHEGAR A UMA ESCRITA ENVOLVENDO UM QUADRADO PERFEITO.

$$z^2-z = (z+a)^2+b$$

$$= z^2+2az+a^2+b$$

$$\begin{cases} 2a = -1 \to a = -\frac{1}{2} \\ a^2+b = 0 \end{cases}$$

$$\left(-\frac{1}{2}\right)^2+b = 0$$

$$b = -\frac{1}{4}$$

$$\Rightarrow z^2-z = \left(z-\frac{1}{2}\right)^2-\frac{1}{4}$$

Disso, obtemos:

$$x^2+y^2+z^2-z = 0$$

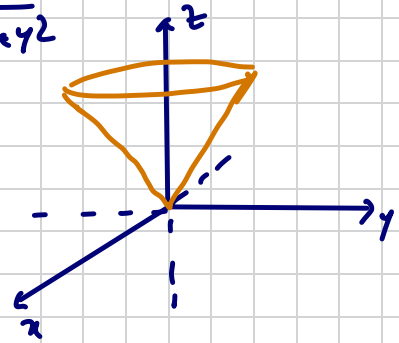
$$x^2+y^2+\left(z-\frac{1}{2}\right)^2-\frac{1}{4} = 0$$

$$x^2 + y^2 + \left(z - \frac{1}{2}\right)^2 = \frac{1}{4}$$

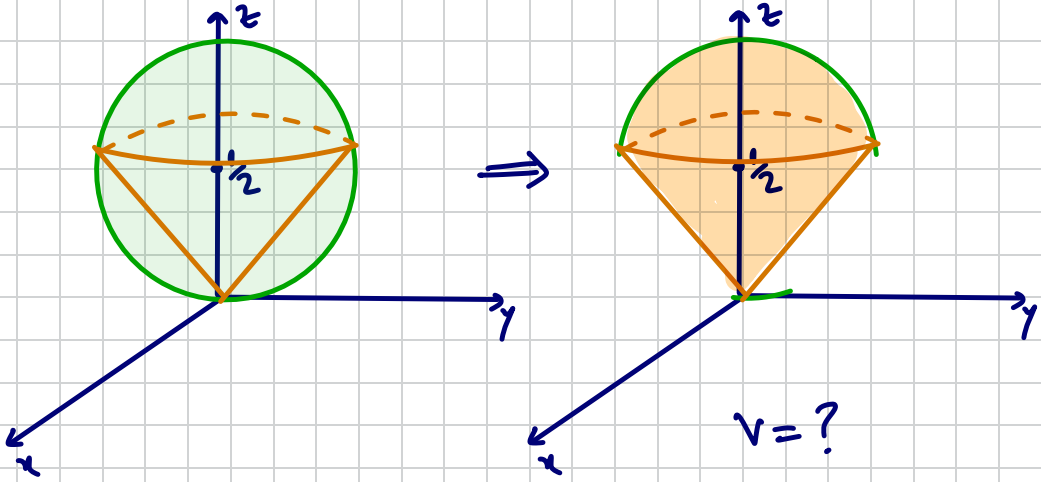
$$(x-0)^2 + (y-0)^2 + \left(z - \frac{1}{2}\right)^2 = \frac{1}{4}$$

↳ circunferência de raio $\frac{1}{2}$ e centrada no ponto $P\left(0, 0, \frac{1}{2}\right)$

cone: $z = \sqrt{x^2 + y^2}$



O intercepto das duas retas:



(CONTINUAMOS NA SEGUNDA)