CÁLCULO 3, TURMA T2

20/07/26 – AULA 46

AULA DE EXERCÍCIOS

LISTA 09:

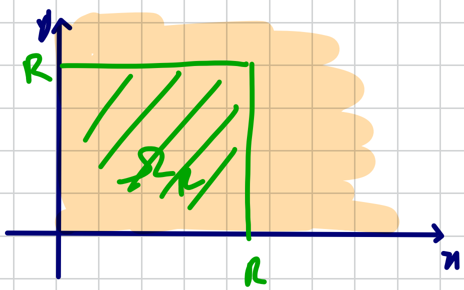
13. Calcule as integrais impróprias abaixo.

(a) $\displaystyle\int_0^\infty \int_0^\infty e^{-x-y}dxdy$ (b) $\displaystyle\int_0^\infty \int_0^\infty x^2 e^{-x^2-y^2}dxdy$

(c) $\displaystyle\iint_{x^2+y^2\leq 1} \frac{x^2 dxdy}{(x^2+y^2)^{\frac{7}{4}}}$ (d) $\displaystyle\iint_{x^2+y^2\leq 1} \ln\sqrt{x^2+y^2}dxdy$

(a) $\displaystyle\int_0^\infty \int_0^\infty e^{-x-y}dxdy$

$\Omega_R = [0,R]\times[0,R]$



Assim, teremos:

$$\int_0^\infty \int_0^\infty e^{-x-y}dxdy = \lim_{R\to\infty} \int_{y=0}^{y=R} \int_{x=0}^{x=R} e^{-x-y}dxdy$$

$$= \lim_{R\to\infty} \int_{y=0}^{y=R} \int_{x=0}^{x=R} e^{-x}\cdot e^{-y}dxdy =$$

$$= \lim_{R\to\infty} \int_{y=0}^{y=R} e^{-y}\,dy \cdot \int_{x=0}^{x=R} e^{-x}\,dx =$$

$$= \lim_{R\to\infty} \left[-\int_{y=0}^{R} e^{-y}\cdot(-1\,dy) \right] \cdot \left[-\int_{x=0}^{x=R} e^{-x}\cdot(-dx) \right]$$

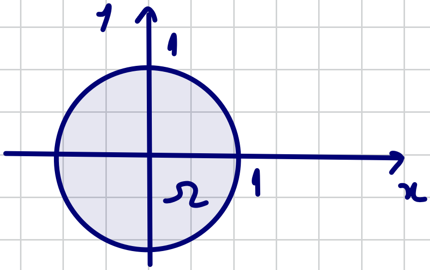
$$= \lim_{R\to\infty} -\left(e^{-y} \right)\Big|_0^R \cdot \left(-e^{-x} \right)\Big|_0^R$$

$$= +\lim_{R\to\infty} \left(e^{-R} - e^{0} \right)\cdot\left(e^{-R} - e^{0} \right) = \lim_{R\to\infty} \left(\underbrace{\frac{1}{e^R}}_{\to 0} - 1 \right)^2$$

$$= (-1)^2 = 1$$

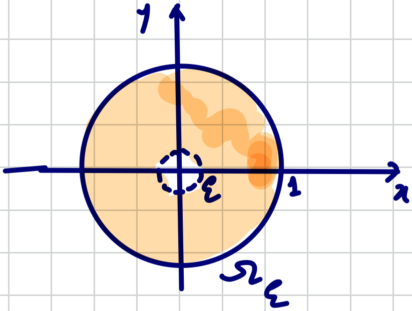
esta função não está definida na origem

(d) $$\iint_{x^2+y^2\le 1} \ln\sqrt{x^2+y^2}\,dx\,dy = \iint_{\Omega} \ln\left(x^2+y^2\right)^{\frac{1}{2}} dx\,dy =$$



$$= \frac{1}{2} \iint_{\Omega} \ln\left(x^2+y^2\right) dx\,dy$$

Tome $\varepsilon > 0$; $0 < \varepsilon < 1$.



$$\Omega_\varepsilon = \Omega \setminus B_\varepsilon(0).$$

Assim, teremos:

$$\frac{1}{2}\iint_{\Omega} \ln(x^2+y^2)\,dx\,dy = \lim_{\varepsilon \to 0^+} \frac{1}{2}\iint_{\Omega_\varepsilon} \ln(x^2+y^2)\,dx\,dy$$

$$= \lim_{\varepsilon \to 0^+} \frac{1}{2} \cdot \int_{\theta=0}^{\theta=2\pi} \int_{\rho=\varepsilon}^{\rho=1} \ln \rho^2 \cdot \rho\, d\rho\, d\theta =$$

COORD. POLARES:

$x^2+y^2 = \rho^2$

$dx\,dy = \rho\, d\rho\, d\theta$

$$= \lim_{\varepsilon \to 0^+} \frac{1}{\cancel{2}} \cdot \int_{\theta=0}^{\theta=2\pi} \int_{\rho=\varepsilon}^{\rho=1} \cancel{2} \cdot \ln \rho \cdot \rho\, d\rho\, d\theta =$$

$$= \lim_{\varepsilon \to 0^+} \int_{\theta=0}^{\theta=2\pi} \underbrace{\int_{\rho=\varepsilon}^{\rho=1} \ln \rho \cdot \rho\, d\rho}_{\text{INT. POR PARTES:}}\, d\theta =$$

$$\int \ln\rho \cdot \rho\, d\rho = \int u\, dv = u \cdot v - \int v\, du$$

$$\begin{cases} u = \ln\rho \Rightarrow du = \dfrac{1}{\rho}\, d\rho \\ dv = \rho\, d\rho \Rightarrow v = \dfrac{\rho^2}{2} \end{cases}$$

$$\Rightarrow \int \ln\rho \cdot \rho\, d\rho = \frac{\rho^2}{2} \cdot \ln\rho - \int \frac{\cancel{\rho^2}}{2} \cdot \frac{d\rho}{\cancel{\rho}}$$

$$= \frac{\rho^2}{2} \cdot \ln\rho - \frac{1}{2} \int \rho\, d\rho$$

$$= \frac{\rho^2}{2} \ln\rho - \frac{1}{2} \frac{\rho^2}{2} + C$$

$$= \frac{\rho^2}{2} \left[\ln\rho - \frac{1}{2} \right] + C$$

$$= \lim_{\varepsilon \to 0^+} \int_{\theta=0}^{\theta=2\pi} \left(\frac{\rho^2}{2} \cdot \left[\ln\rho - \frac{1}{2} \right] \right) \Bigg|_{\rho=\varepsilon}^{\rho=1} \cdot d\theta \,.$$

$$= \lim_{\varepsilon \to 0^+} \int_{\theta=0}^{\theta=2\pi} d\theta \cdot \left[\frac{1}{2} \Big(\underbrace{\ln 1}_{\substack{\shortparallel \\ 0}} - \frac{1}{2} \Big) - \frac{\varepsilon^2}{2} \Big(\ln\varepsilon - \frac{1}{2} \Big) \right]$$

$$= \theta \Big|_0^{2\pi} \cdot \lim_{\varepsilon \to 0^+} \Big(-\frac{1}{4} - \underbrace{\frac{\varepsilon^2}{2} \cdot \ln\varepsilon}_{\text{INDETERMINAÇÃO } 0 \cdot \infty} - \frac{\varepsilon^2}{2} \Big)$$

$$= 2\pi \cdot \lim_{\varepsilon\to 0^+}\left(-\frac{1}{4} - \frac{1}{2}\cdot\underbrace{\overbrace{\frac{\ln\varepsilon}{\frac{1}{\varepsilon^2}}}^{=0}}_{\frac{\infty}{\infty}\ \text{L'HOPITAL}} - \frac{\overset{0}{\cancel{\varepsilon^2}}}{2}\right)$$

$$= 2\pi\cdot\left(-\frac{1}{4}\right) = -\frac{\pi}{2}$$

L9

6. Calcule $\displaystyle\iint_\Omega \arctan\frac{y}{x}\,dA$, onde Ω é a região do primeiro quadrante compreendida entre dois círculos $x^2+y^2=4$ e $x^2+y^2=2x$.

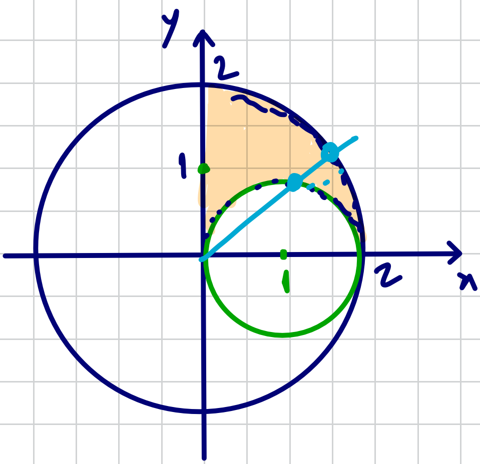
$$\underbrace{x^2+y^2=2x}$$

$$x^2-2x+y^2=0$$

$$x^2-2x+1+y^2=1$$

$$(x-1)^2+y^2=1$$

$\hookrightarrow$ circunferência de raio unitário centrada em $(1,0)$



$$0\le\theta\le\frac{\pi}{2}$$

$$\left\{\begin{array}{l}\underbrace{\text{PONTOS DA CIRCONF. MENOR}} \le \rho \le 2 \\ \underline{x^2+y^2=2x} \\ \rho^{\cancel{2}} = 2\cdot\cancel{\rho}\cos\theta \Rightarrow \rho = 2\cos\theta\end{array}\right.$$

$$2\cos\theta \le \rho \le 2$$

$$\tan\theta = \frac{y}{x} \Rightarrow \theta = \arctan\frac{y}{x} \quad ; \quad dx\,dy = \rho\,d\rho\,d\theta$$

Assim, temos:

$$\iint_{\Omega} \arctan\frac{y}{x}\,.dx\,dy = \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{\rho=2\cos\theta}^{\rho=2} \theta\,.\,\rho\,.d\rho\,d\theta =$$

$$= \int_{\theta=0}^{\theta=\frac{\pi}{2}} \theta\,.\left(\int_{\rho=2\cos\theta}^{\rho=2} \rho\,d\rho\right)d\theta = \int_{\theta=0}^{\theta=\frac{\pi}{2}} \theta\,.\,\left.\frac{\rho^2}{2}\right|_{\rho=2\cos\theta}^{\rho=2} d\theta =$$

$$= \int_{\theta=0}^{\theta=\frac{\pi}{2}} \theta\,.\left(2 - 2\cos^2\theta\right)d\theta =$$

$$= \int_0^{\frac{\pi}{2}} 2\theta\,d\theta - \int_0^{\frac{\pi}{2}} 2\cos^2\theta\,.\,\theta\,d\theta$$

$$= \left.\theta^2\right|_0^{\frac{\pi}{2}} - \underbrace{2\int_0^{\frac{\pi}{2}}.\cos^2\theta\,.\,\theta\,d\theta\,.}_{\text{integrar por partes}} =$$

$$\int \cos^2\theta \cdot \theta \, d\theta = \int u \, dv = u \cdot v - \int v \, du \; \boxed{=}$$

$$u = \theta \Longrightarrow du = d\theta.$$

$$dv = \cos^2\theta \, d\theta \Rightarrow v = \int \cos^2\theta \, d\theta = \int \frac{1+\cos 2\theta}{2} \, d\theta$$

$$= \frac{1}{2}\int d\theta + \frac{1}{2}\frac{1}{2}\int \cos 2\theta \, 2 \, d\theta$$

$$= \frac{\theta}{2} + \frac{1}{4} \cdot \operatorname{sen} 2\theta$$

$$= \theta \cdot \left(\frac{\theta}{2} + \frac{1}{4}\operatorname{sen} 2\theta\right) - \int \left(\frac{\theta}{2} + \frac{1}{4}\operatorname{sen} 2\theta\right) d\theta$$

$$= \frac{\theta^2}{2} + \frac{\theta}{4} \cdot \operatorname{sen} 2\theta - \frac{1}{2}\int \theta \, d\theta - \frac{1}{4}\int \operatorname{sen} 2\theta \, d\theta$$

$$= \cancel{\frac{\theta^2}{2}} + \frac{\theta}{4}\operatorname{sen} 2\theta - \cancel{\frac{\theta^2}{2}} + \frac{1}{8}\cos 2\theta + C$$

$$= \frac{\theta}{4} \cdot \operatorname{sen} 2\theta + \frac{1}{8}\cos 2\theta + C$$

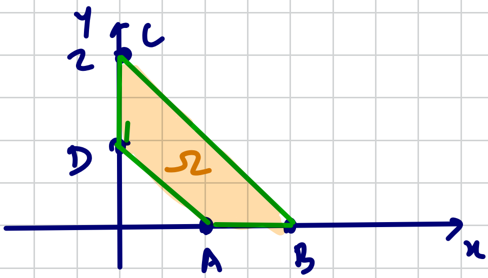
$$= \left(\frac{\pi}{2}\right)^2 - 2 \cdot \left(\frac{\theta}{4}\operatorname{sen} 2\theta + \frac{1}{8}\cos 2\theta\right)\Bigg|_0^{\frac{\pi}{2}} =$$

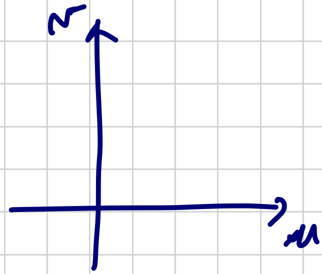
$$= \frac{\pi^2}{4} - 2 \cdot \left(\frac{\pi}{8}\underbrace{(\operatorname{sen}\pi - \operatorname{sen} 0)}_{=0} + \frac{1}{8}(\underbrace{\cos\pi}_{-1} - \cos 0)\right)$$

$$= \frac{\pi^2}{4} - 2\cdot\left(0 + \frac{1}{8}\cdot(-2)\right) = \frac{\pi^2}{4} + \frac{1}{2}$$

L9

11. Calcule $\displaystyle\iint_{\Omega} \cos\left(\frac{y-x}{y+x}\right) dA$, onde Ω é a região trapezoidal com vértices em $A(1,0)$, $B(2,0)$, $C(0,2)$ e $D(0,1)$.





Escreva $\begin{cases} u = y - x \\ v = y + x \end{cases}$

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$$u + v = 2y \Rightarrow \boxed{y = \frac{1}{2}u + \frac{1}{2}v}$$

$$v = y + x \Rightarrow x = v - y$$

$$x = v - \left(\frac{1}{2}u + \frac{1}{2}v\right)$$

$$\boxed{x = -\frac{1}{2}u + \frac{1}{2}v}$$

$$\Rightarrow \iint_{\Omega} \cos\left(\frac{y-x}{y+x}\right) dx\,dy = \iint_{\Omega'} \cos\frac{u}{v} \cdot \left|\det J(T)(u,v)\right| . du . dv ,$$

onde:

$$J(T)(u,v) = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\Rightarrow \det J(T)(u,v) = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

$-$ $+$

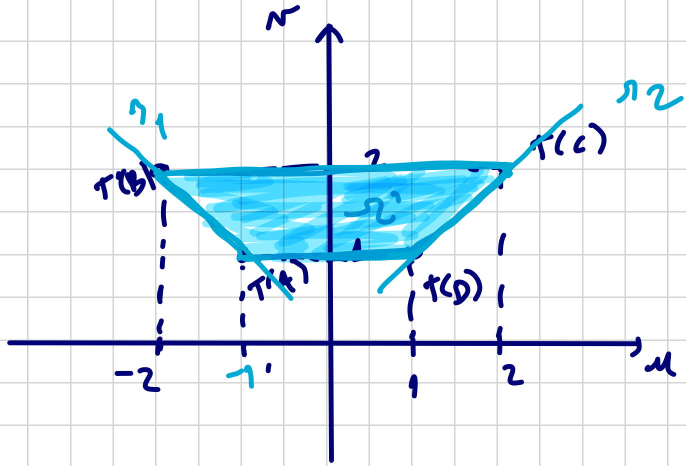
$$\Rightarrow \left|\det(J(T)(u,v)\right| = \left|-\frac{1}{2}\right| = \frac{1}{2} .$$

Disso, teremos:

$$\iint_{\Omega} \cos\left(\frac{y-x}{y+x}\right) dx\,dy = \iint_{\Omega'} \cos\frac{u}{v} \cdot \frac{1}{2} . du\,dv =$$

$$= \frac{1}{2} \iint_{\Omega'} \cos\frac{u}{v} . du\,dv =$$

(x,y)	$(u,v) = (y-x,\ y+x)$
$A(1,0)$	$T(A) = (0-1,\ 0+1) = (-1,1)$
$B(2,0)$	$T(B) = (0-2,\ 0+2) = (-2,2)$
$C(0,2)$	$T(C) = (2-0,\ 2+0) = (2,2)$
$D(0,1)$	$T(D) = (1-0,\ 1+0) = (1,1)$



$r_1:\ v = -u$

$r_2:\ v = u$

$$= \frac{1}{2}\int_{v=1}^{v=2}\int_{u=r_1}^{u=r_2} \cos\frac{u}{v}\,.\,du\,dv = \frac{1}{2}\int_{v=1}^{v=2}\int_{u=-v}^{u=v} \cos\frac{u}{v}\,du\,dv$$

$$= \frac{1}{2}\int_{v=1}^{v=2}\int_{u=-v}^{u=v} \cos\frac{u}{v}\cdot\frac{1}{v}\,.\,v\,du\,dv =$$

$$\int \cos\frac{u}{v}\,du \qquad w = \frac{u}{v} \qquad dw = \frac{1}{v}\,du$$

$$= \frac{1}{2} \int_{r=1}^{r=2} r \left(\int_{u=-r}^{u=r} \cos\left(\frac{u}{r}\right) \cdot \frac{du}{r} \right) dr$$

$$= \frac{1}{2} \int_{r=1}^{r=2} r \cdot \left(\operatorname{sen} \frac{u}{r} \right) \Bigg|_{u=-r}^{u=r} dr =$$

$$= \frac{1}{2} \int_{r=1}^{r=2} r \cdot \left(\operatorname{sen} 1 - \operatorname{sen}(-1) \right) dr =$$

$$= \frac{1}{2} \int_{r=1}^{r=2} r \cdot \left(\operatorname{sen} 1 + \operatorname{sen} 1 \right) \cdot dr =$$

$$= \frac{1}{\cancel{2}} \cdot \cancel{2} \cdot \operatorname{sen} 1 \cdot \int_{r=1}^{r=2} r \, dr$$

$$= \operatorname{sen} 1 \cdot \frac{r^2}{2} \Bigg|_1^2 = \operatorname{sen} 1 \cdot \left(2 - \frac{1}{2} \right) = \frac{3}{2} \cdot \operatorname{sen} 1$$