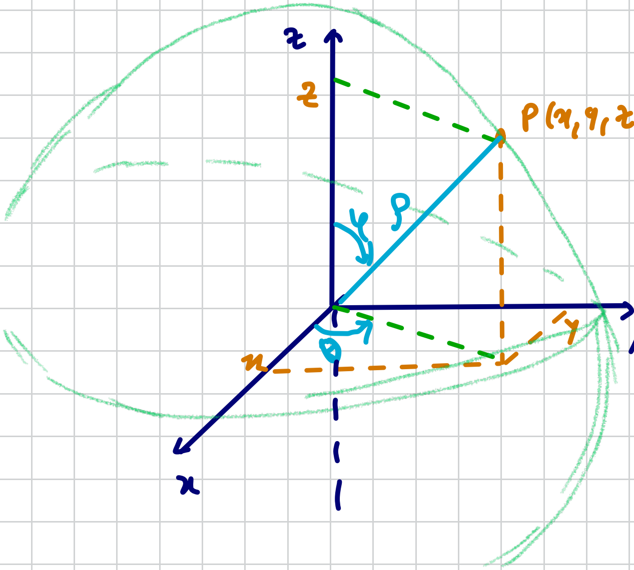
CÁLCULO 3, TURMA T2

02/07/26 – AULA 38

SISTEMA DE COORDENADAS ESFÉRICAS Neste sistema, um ponto $P(x,y,z)$ fica identificado como sendo um ponto sobre uma esfera.

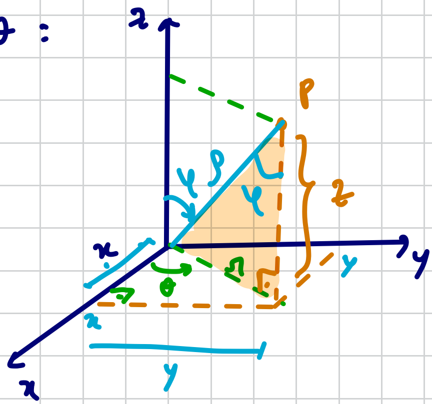


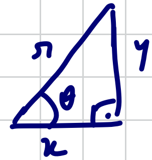
ρ : raio ; $\rho \in \mathbb{R}$

$0 \leq \theta \leq 2\pi$ (no plano xy)

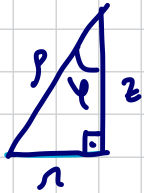
$0 \leq \varphi \leq \pi$ (partindo do semi-eixo positivo z até o semi-eixo negativo z)

Relação entre os sistemas retangular xyz e o esférico $\rho\varphi\theta$:





$$\begin{cases} \operatorname{sen}\theta = \dfrac{y}{r} \Rightarrow y = r\operatorname{sen}\theta \\ \cos\theta = \dfrac{x}{r} \Rightarrow x = r\cos\theta \end{cases}$$



$$\operatorname{sen}\varphi = \frac{r}{\rho} \Rightarrow \boxed{r = \rho \operatorname{sen}\varphi}$$

$$\cos\varphi = \frac{z}{\rho} \Rightarrow \boxed{z = \rho\cos\varphi}$$

Assim, obtemos:

$$x = r\cdot\cos\theta = \rho\cdot\operatorname{sen}\varphi\cdot\cos\theta$$

$$y = r\cdot\operatorname{sen}\theta = \rho\cdot\operatorname{sen}\varphi\cdot\operatorname{sen}\theta.$$

$$z = \rho\cdot\cos\varphi$$

Ou seja, do sistema xyz para o sistema esférico $\rho\varphi\theta$, temos a transformação T dada por:

$$T: \begin{cases} x = \rho\operatorname{sen}\varphi\cos\theta \\ y = \rho\operatorname{sen}\varphi\cdot\operatorname{sen}\theta \\ z = \rho\cos\varphi \end{cases}$$

Neste caso, no sistema esférico teremos:

$$x^2+y^2+z^2 = (\rho\operatorname{sen}\varphi\cos\theta)^2 + (\rho\operatorname{sen}\varphi\operatorname{sen}\theta)^2 + (\rho\cos\varphi)^2$$

$$= \rho^2\cdot\operatorname{sen}^2\varphi\cdot\cos^2\theta + \rho^2\operatorname{sen}^2\varphi\cdot\operatorname{sen}^2\theta + \rho^2\cos^2\varphi$$

$$= \rho^2\operatorname{sen}^2\varphi\cdot(\underbrace{\cos^2\theta + \operatorname{sen}^2\theta}_{=1}) + \rho^2\cos^2\varphi$$

$$= \rho^2 \operatorname{sen}^2\varphi + \rho^2\cos^2\varphi = \rho^2 \underbrace{(\operatorname{sen}^2\varphi + \cos^2\varphi)}_{=1} = \rho^2$$

$$\Longrightarrow \boxed{x^2+y^2+z^2=\rho^2}$$

Isto justifica o nome: sistema de coordenadas esféricas.

Dado $f: \Omega \subset \mathbb{R}^3 \to \mathbb{R}$ uma função integrável, onde Ω ou parte de Ω é dada por uma esfera, então:

$$\iiint_{\Omega} f(x,y,z)\,dV =$$

$$= \iiint_{\Omega'} f(x(\rho,\varphi,\theta), y(\rho,\varphi,\theta), z(\rho,\varphi,\theta)) \cdot \underbrace{|\det J(T)(\rho,\varphi,\theta)|}_{?}\, d\rho\, d\varphi\, d\theta.$$

Precisamos determinar $\det J(T)(\rho,\varphi,\theta)$. Como

$$T: \begin{cases} x = \rho \operatorname{sen}\varphi \cos\theta \\ y = \rho \operatorname{sen}\varphi \operatorname{sen}\theta \\ z = \rho\cos\varphi \end{cases}, \quad \text{então}$$

a matriz Jacobiana dessa transformação será:

$$J(T)(\rho,\varphi,\theta) = \begin{bmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial y}{\partial \rho} & \frac{\partial z}{\partial \rho} \\ \frac{\partial x}{\partial \varphi} & \frac{\partial y}{\partial \varphi} & \frac{\partial z}{\partial \varphi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \end{bmatrix} =$$

$$= \begin{bmatrix} \operatorname{sen}\varphi \cdot \cos\theta & \operatorname{sen}\varphi \cdot \operatorname{sen}\theta & \cos\varphi \\ \rho \cdot \cos\varphi \cdot \cos\theta & \rho\cos\varphi\operatorname{sen}\theta & -\rho\operatorname{sen}\varphi \\ -\rho\operatorname{sen}\varphi\operatorname{sen}\theta & \rho\operatorname{sen}\varphi\cos\theta & 0 \end{bmatrix} .$$

Disso:

$$\det J(T)(\rho,\varphi,\theta) = \begin{vmatrix} \operatorname{sen}\varphi \cdot \cos\theta & \operatorname{sen}\varphi \cdot \operatorname{sen}\theta & \cos\varphi \\ \rho \cdot \cos\varphi \cdot \cos\theta & \rho\cos\varphi\operatorname{sen}\theta & -\rho\operatorname{sen}\varphi \\ -\rho\operatorname{sen}\varphi\operatorname{sen}\theta & \rho\operatorname{sen}\varphi\cos\theta & 0 \end{vmatrix}$$

$$= \left| \begin{matrix} \operatorname{sen}\varphi \cdot \cos\theta & \operatorname{sen}\varphi \cdot \operatorname{sen}\theta & \cos\varphi \\ \rho \cdot \cos\varphi \cdot \cos\theta & \rho\cos\varphi\operatorname{sen}\theta & -\rho\operatorname{sen}\varphi \\ -\rho\operatorname{sen}\varphi\operatorname{sen}\theta & \rho\operatorname{sen}\varphi\cos\theta & 0 \end{matrix} \right| \begin{matrix} \operatorname{sen}\varphi \cdot \cos\theta & \operatorname{sen}\varphi \cdot \operatorname{sen}\theta \\ \rho \cdot \cos\varphi \cdot \cos\theta & \rho\cos\varphi\operatorname{sen}\theta \\ -\rho\operatorname{sen}\varphi\operatorname{sen}\theta & \rho\operatorname{sen}\varphi\cos\theta \end{matrix}$$

$$\begin{matrix} - & - & - & + & + & + \end{matrix}$$

$$= 0 + \rho^2\operatorname{sen}^3\varphi\operatorname{sen}^2\theta + \rho^2\operatorname{sen}\varphi\cos^2\varphi\cos^2\theta +$$

$$+ \rho^2 \cdot \operatorname{sen}\varphi \cdot \cos^2\varphi \cdot \operatorname{sen}^2\theta + \rho^2 \cdot \operatorname{sen}^3\varphi \cdot \cos^2\theta - 0$$

$$= \rho^2 \cdot \operatorname{sen}^3\varphi \, (\underbrace{\operatorname{sen}^2\theta + \cos^2\theta}_{=1}) + \rho^2 \operatorname{sen}\varphi \cos^2\varphi \, (\underbrace{\cos^2\theta + \operatorname{sen}^2\theta}_{=1})$$

$$= \rho^2 \operatorname{sen}^3\varphi + \rho^2 \operatorname{sen}\varphi \cos^2\varphi =$$

$$= \rho^2 \operatorname{sen}\varphi \, (\underbrace{\operatorname{sen}^2\varphi + \cos^2\varphi}_{=1}) = \rho^2 \cdot \operatorname{sen}\varphi$$

$$\Rightarrow \det J(T)(\rho, \varphi, \theta) = \rho^2 \operatorname{sen}\varphi$$

$$\Rightarrow |\det J(T)(\rho, \varphi, \theta)| = |\rho^2 \operatorname{sen}\varphi| = \rho^2 \cdot \operatorname{sen}\varphi$$

$$0 \leq \varphi \leq \pi$$

Assim, concluímos que:

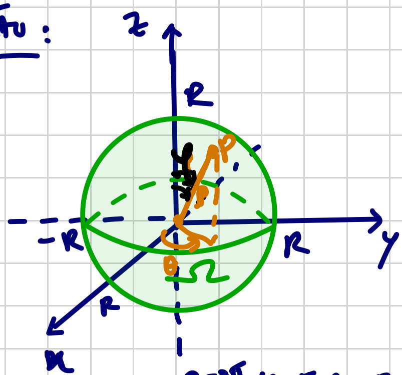
$$\boxed{\begin{aligned} &\iiint_{\Omega} f(x,y,z)\, dx\, dy\, dz = \\ &\quad = \iiint_{\Omega'} f(x(\rho,\varphi,\theta), y(\rho,\varphi,\theta), z(\rho,\varphi,\theta)) \cdot \rho^2 \operatorname{sen}\varphi \cdot d\rho\, d\varphi\, d\theta. \end{aligned}}$$

FÓRMULA DE MUDANÇA DO SISTEMA RETANGULAR PARA O SISTEMA ESFÉRICO

No que segue apresentamos alguns exemplos:

01) Usando coordenadas esféricas, mostre que a fórmula do volume de uma esfera de raio $R>0$ é dada por $V = \frac{4\pi R^3}{3}$.

Solução:



$0 \leq \rho \leq R$

$$V = \iiint_\Omega dV = \int_{\theta=0}^{\theta=2\pi} \int_{\varphi=0}^{\varphi=\pi} \int_{\rho=0}^{\rho=R} \rho^2 \operatorname{sen}\varphi \, d\rho \, d\varphi \, d\theta =$$

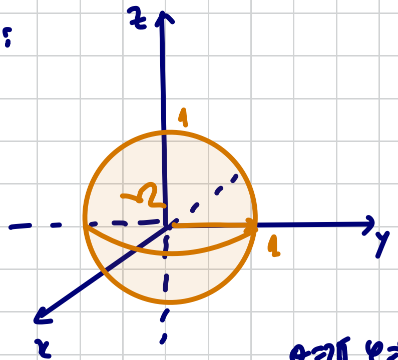
$$= \int_{\theta=0}^{\theta=2\pi} d\theta \cdot \int_{\varphi=0}^{\varphi=\pi} \operatorname{sen}\varphi \, d\varphi \cdot \int_{\rho=0}^{\rho=R} \rho^2 d\rho =$$

$$= \theta \Big|_0^{2\pi} (-\cos\varphi) \Big|_0^{\pi} \cdot \frac{\rho^3}{3} \Big|_0^{R} =$$

$$= 2\pi \cdot \underbrace{(\underbrace{-\cos\pi}_{-1} + \underbrace{\cos 0}_{1})}_{2} \cdot \frac{R^3}{3} = 2\pi \cdot 2 \cdot \frac{R^3}{3} = \frac{4\pi R^3}{3}$$

02) Calcule $\iiint_\Omega e^{(x^2+y^2+z^2)^{\frac{3}{2}}} \cdot dV$, sendo Ω a região dada por: $\Omega = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \leq 1\}$.

Solução:



$$\boxed{x^2+y^2+z^2 = \rho^2}$$

$$\iiint_\Omega e^{(x^2+y^2+z^2)^{\frac{3}{2}}} dV = \int_{\theta=0}^{\theta=2\pi}\int_{\varphi=0}^{\varphi=\pi}\int_{\rho=0}^{\rho=1} e^{(\rho^{\cancel{2}})^{\frac{3}{\cancel{2}}}} \rho^2 \cdot \operatorname{sen}\varphi \cdot d\rho\, d\varphi\, d\theta =$$

$$= \int_{\theta=0}^{\theta=2\pi} d\theta \cdot \int_{\varphi=0}^{\varphi=\pi} \operatorname{sen}\varphi \cdot d\varphi \cdot \frac{1}{3}\underbrace{\int_{\rho=0}^{\rho=1} e^{\overbrace{\rho^3}^{u}} \underbrace{(3\rho^2 d\rho)}_{du}} =$$

$\hookrightarrow \int e^u du = e^u + C$

$u = \rho^3 \rightarrow du = 3\rho^2 d\rho$

$$= \theta\Big|_0^{2\pi} (-\cos\varphi)\Big|_0^{\pi} \cdot \frac{1}{3} \cdot e^{\rho^3}\Big|_0^1 =$$

$$= 2\pi \cdot (\underbrace{\underbrace{-\cos\pi}_{-1} + \underbrace{\cos 0}_{1}}_{+2}) \cdot \frac{1}{3} \cdot (e^1 - \underbrace{e^0}_{=1})$$

$$= 2\pi \cdot (2) \cdot \frac{1}{3} \cdot (e-1) = \frac{4\pi}{3} \cdot (e-1)$$

03) Use coordenadas esféricas para determinar o volume do sólido acima do cone $z = \sqrt{x^2+y^2}$ e abaixo da esfera $x^2+y^2+z^2 = z$.

Solução: Primeiramente, note que a esfera não está centrada na origem. Iremos completar um quadrado perfeito para obter o centro e o raio da esfera.

$$x^2+y^2+\underbrace{z^2-z}=0$$

$$z^2 - z = (z+a)^2 + b$$

$$= z^2 + 2az + a^2 + b$$

$$\begin{cases} 2a = -1 \rightarrow \boxed{a = -\frac{1}{2}} \\ a^2+b=0 \rightarrow \left(-\frac{1}{2}\right)^2 + b = 0 \quad \boxed{b = -\frac{1}{4}} \end{cases}$$

Dica: $z^2 - z = \left(z - \frac{1}{2}\right)^2 - \frac{1}{4}$

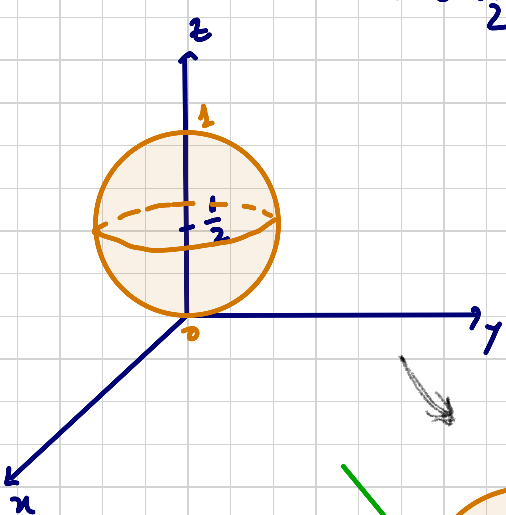
Então, temos:

$$x^2 + y^2 + z^2 - z = 0$$

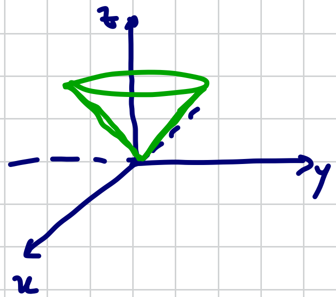
$$\Leftrightarrow \quad x^2 + y^2 + \left(z - \frac{1}{2}\right)^2 - \frac{1}{4} = 0$$

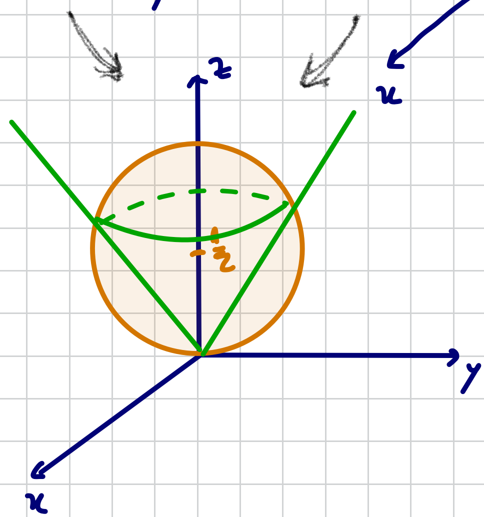
$$\Leftrightarrow \quad x^2 + y^2 + \left(z - \frac{1}{2}\right)^2 = \frac{1}{4}$$

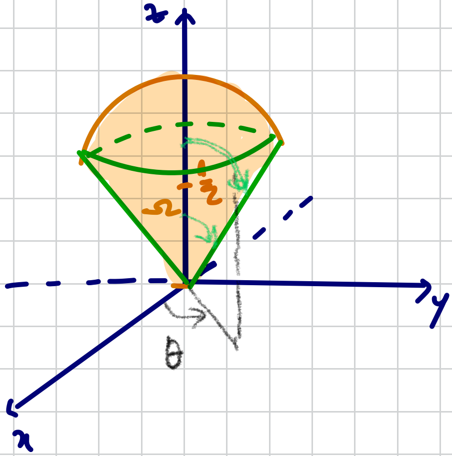
↳ circunferência centrada em $P\left(0, 0, \frac{1}{2}\right)$ e raio $\frac{1}{2}$



cone: $z = \sqrt{x^2 + y^2}$







$$V = \iiint_{\Omega} dV$$

$0 \leq \theta \leq 2\pi$

$0 \leq \rho \leq$ PONTOS SOBRE A ESFERA.

$0 \leq \varphi \leq$ PONTOS SOBRE A CASCA DO CONE.

Para φ: até o cone, temos:

$$z = \sqrt{x^2+y^2} \qquad \begin{cases} x = \rho \operatorname{sen}\varphi \cos\theta \\ y = \rho \operatorname{sen}\varphi \operatorname{sen}\theta \\ z = \rho \cos\varphi \end{cases}$$

$$\rho\cos\varphi = \sqrt{(\rho \operatorname{sen}\varphi \cos\theta)^2 + (\rho \operatorname{sen}\varphi \operatorname{sen}\theta)^2}$$

$$\rho\cos\varphi = \sqrt{\rho^2 \operatorname{sen}^2\varphi \cos^2\theta + \rho^2 \operatorname{sen}^2\varphi \operatorname{sen}^2\theta}$$

$$\rho\cos\varphi = \sqrt{\rho^2 \operatorname{sen}^2\varphi \, (\underbrace{\cos^2\theta + \operatorname{sen}^2\theta}_{=1})}$$

$$\rho\cos\varphi = \sqrt{\rho^2 \operatorname{sen}^2\varphi} = \rho \cdot \operatorname{sen}\varphi$$

$0 \leq \varphi \leq \pi \to \operatorname{sen}\varphi \geq 0$

$$\Rightarrow \cancel{\rho}\cos\varphi = \cancel{\rho}\operatorname{sen}\varphi \Rightarrow \cos\varphi = \operatorname{sen}\varphi$$

$$\Leftrightarrow \boxed{\varphi = \frac{\pi}{4}}$$

Logo, concluímos que devemos ter $0 < \varphi \le \frac{\pi}{4}$.

Resta encontrar variação para ρ.

$0 < \rho \le$ PONTOS SOBRE A ESFERA.

$$\underbrace{x^2+y^2+z^2}_{\rho^2} = \underbrace{z}_{=\rho\cos\varphi}$$

$$\Rightarrow \rho^{\cancel{2}} = \cancel{\rho}\cos\varphi \Rightarrow \rho = \cos\varphi$$

Portanto, obtemos $\boxed{0 \le \rho \le \cos\varphi}$

Com isso, segue que:

$$V = \iiint_{\Omega} dV = \int_{\theta=0}^{\theta=2\pi}\int_{\varphi=0}^{\varphi=\frac{\pi}{4}}\int_{\rho=0}^{\rho=\cos\varphi} \rho^2 \cdot \operatorname{sen}\varphi \, d\rho \, d\varphi \, d\theta \cdot =$$

$$= \int_{\theta=0}^{\theta=2\pi} d\theta \int_{\varphi=0}^{\varphi=\frac{\pi}{4}} \operatorname{sen}\varphi \Big(\underbrace{\int_{\rho=0}^{\rho=\cos\varphi} \rho^2 \, d\rho}_{g(\varphi)} \Big) d\varphi =$$

$$= \theta \Big|_0^{2\pi} \cdot \int_{\varphi=0}^{\varphi=\frac{\pi}{4}} \operatorname{sen}\varphi \cdot \frac{\rho^3}{3} \Big|_0^{\cos\varphi} d\varphi =$$

$$= 2\pi \cdot \int_{\varphi=0}^{\varphi=\frac{\pi}{4}} \operatorname{sen}\varphi \cdot \left(\frac{\cos^3\varphi}{3} - 0 \right) d\varphi =$$

$$= \frac{2\pi}{3} \int_{\varphi=0}^{\varphi=\frac{\pi}{4}} \cos^3\varphi \cdot \operatorname{sen}\varphi \, d\varphi =$$

$$= -\frac{2\pi}{3} \int_0^{\frac{\pi}{4}} (\underbrace{\cos\varphi}_{u})^3 \cdot (-\operatorname{sen}\varphi \, d\varphi) =$$

$$\int u^3 \, du = \frac{u^4}{4} + C$$

$$u = \cos\varphi \Rightarrow du = -\operatorname{sen}\varphi \, d\varphi$$

$$= -\frac{\overset{1}{\cancel{2}}\pi}{3} \cdot \left. \frac{(\cos\varphi)^4}{\underset{2}{\cancel{4}}} \right|_0^{\frac{\pi}{4}} = -\frac{\pi}{6} \cdot \left(\cos^4\frac{\pi}{4} - \cos^4 0 \right)$$

$$= -\frac{\pi}{6} \cdot \left(\left(\frac{\sqrt{2}}{2}\right)^4 - (1)^4 \right) = -\frac{\pi}{6} \cdot \left(\frac{4}{16} - 1 \right) =$$

$$= -\frac{\pi}{6} \cdot \left(\frac{1}{4} - 1 \right) = -\frac{\pi}{\underset{2}{\cancel{6}}} \cdot \left(-\frac{\overset{1}{\cancel{3}}}{4} \right) = +\frac{\pi}{8} \text{ u.v.}$$