CÁLCULO 3, TURMA T2:

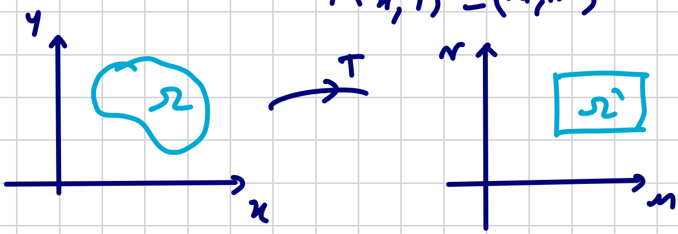
$\boxed{19/06/26 \text{ – AULA } 33}$

Na aula passada estudamos a fórmula de mudança geral de variáveis para integrais duplas:

$$\iint_{\Omega} f(x,y)\cdot dA = \iint_{\Omega'} f(x(u,v),y(u,v)).\,|\det J(T)(u,v)|\,.du\,dv$$

$$T:\Omega \to \Omega' = T(\Omega)$$

$$T(x,y) = (u,v)$$



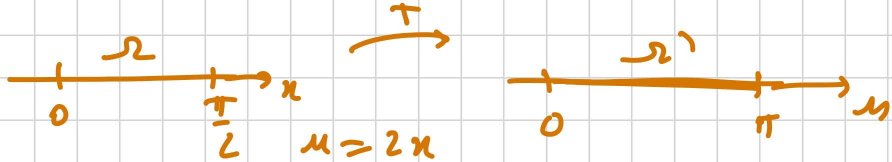
Na verdade, este resultado já era usado no Cálculo 2, sem ser apresentado formalmente.

Ex.: $\displaystyle\int_0^{\frac{\pi}{2}} \operatorname{sen} 2x\, dx = ?$

$T: u = 2x \to x = \frac{1}{2}u$

$J(T)(u) = \left[\frac{\partial x}{\partial u}\right] = \frac{1}{2}$

$\Rightarrow \det(J(T)(u)) = \frac{1}{2}.$



$$u = 2x$$

$$x = \frac{1}{2}u$$

$$\begin{cases} x = 0 \Rightarrow u = 0 \\ x = \frac{\pi}{2} \Rightarrow u = \cancel{2}\cdot\frac{\pi}{\cancel{2}} = \pi \end{cases}$$

$$\int_{x=0}^{x=\frac{\pi}{2}} \operatorname{sen} 2x \, . dx = \int_{u=0}^{u=\pi} \operatorname{sen} u \, . \left|\det(J(T)(u))\right| . du$$

$$= \int_{u=0}^{u=\pi} \operatorname{sen} u \, . \left(\frac{1}{2} du\right)$$

$$= \frac{1}{2}\int_0^{\pi} \operatorname{sen} u \, du = -\frac{1}{2}\cos u \Big|_0^{\pi} =$$

$$= -\frac{1}{2} . \left(\underbrace{\cos \pi}_{-1} - \underbrace{\cos 0}_{1}\right) = -\frac{1}{2}(-1-1) = 1$$

Seguindo nos exemplos:

02) Calcule $\iint_{\Omega} (x-y)^2 \cdot \ln(x+y)\, dx\, dy$, onde Ω é o quadrilátero com vértices nos pontos $A(2,-1)$; $B(4,1)$; $C(2,3)$ e $D(0,1)$.

Solução: Note que, olhando o problema, não temos como separar as variáveis x e y. Isto já sugere uma mudança de variável.

Escreve
$$\begin{cases} u = x - y \\ v = x + y \end{cases}$$
$$+ \; \overline{u + v = 2x} \Rightarrow \boxed{x = \frac{1}{2}u + \frac{1}{2}v}$$

$$y = v - x$$
$$y = v - \frac{1}{2}u - \frac{1}{2}v \Rightarrow \boxed{y = -\frac{1}{2}u + \frac{1}{2}v}$$

ou seja, encontramos

$$T(x,y) = (u,v)$$

$J(T)(u,v) = ?$

$$J(T)(u,v) = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\Rightarrow \det J(T)(u,v) = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{4} - \left(-\frac{1}{4}\right) = \frac{1}{2}$$

$-$ $+$

Assim:

$$\iint_{\Omega} \underbrace{(x-y)}_{u}^2 \ln \underbrace{(x+y)}_{v} \, dx\,dy = \iint_{\Omega'} u^2 . \ln v . \underbrace{|\det(J)(T)(u,v)|}_{\frac{1}{2}} . du\,dv$$

$$= \iint_{\Omega'} u^2 . \ln v . \frac{1}{2} . du\,dv = \frac{1}{2} \iint_{\Omega'} u^2 . \ln v \, du\,dv$$

Resta desenhar a região Ω', a partir de Ω e de T.

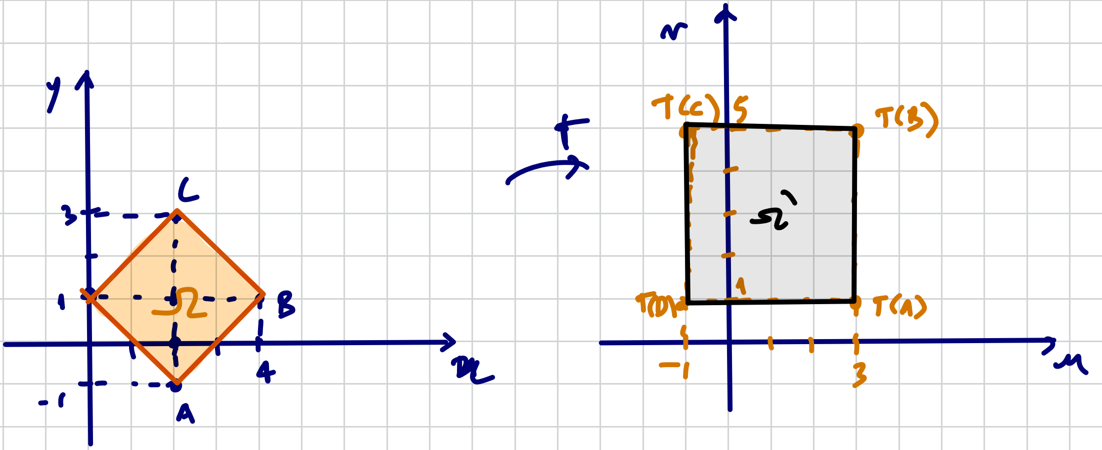
$$(x,y) \xmapsto{T} (u,v) = (x-y,\, x+y)$$

$$A(2,-1) \xmapsto{T} T(A) = (2-(-1),\, 2+(-1)) = (3,1)$$

$$B(4,1) \xmapsto{T} T(B) = (4-1,\, 4+1) = (3,5)$$

$$C(2,3) \xmapsto{T} T(C) = (2-3,\ 2+3) = (-1,5)$$

$$D(0,1) \xmapsto{T} T(D) = (0-1,\ 0+1) = (-1,1)$$



$$-1 \le u \le 3$$

$$1 \le v \le 5.$$

Assim, obtemos:

$$\iint_{\Omega} (x-y)^2 \ln(x+y)\,dx\,dy = \frac{1}{2}\iint_{\Omega'} u^2 \ln v\,du\,dv =$$

$$= \frac{1}{2}\int_{v=1}^{v=5}\int_{u=-1}^{u=3} u^2 \ln v\,du\,dv = \frac{1}{2}\underbrace{\int_{v=1}^{v=5} \ln v\,dv}_{\text{integrar por partes.}} \cdot \int_{u=-1}^{u=3} u^2\,du =$$

$$\bullet \int \ln r \, dr = \int u \, dw = u \cdot w - \int w \, du =$$

$$\left[\begin{array}{l} u = \ln r \Rightarrow du = \dfrac{dr}{r} \\ dw = dr \Rightarrow w = r \end{array}\right.$$

$$= \ln r \cdot r - \int \cancel{r} \cdot \frac{dr}{\cancel{r}} = r \cdot \ln r - \int dr$$

$$= r \cdot \ln r - r + c$$

$$= \frac{1}{2} \left(r \cdot \ln r - r\right)\Big|_{r=1}^{r=5} \cdot \left(\frac{u^3}{3}\right)\Big|_{u=-1}^{u=3} =$$

$$= \frac{1}{2} \cdot \left(5 \cdot \ln 5 - 5 - [\underbrace{1 \cdot \ln 1}_{0} - 1]\right) \cdot \left(\frac{27}{3} - \frac{(-1)}{3}\right)$$

$$= \frac{1}{\cancel{2}} \cdot \left(5 \ln 5 - 5 + 1\right) \cdot \frac{\cancel{28}^{14}}{3} = \frac{14}{3} \cdot \left(5 \cdot \ln 5 - 4\right)$$

03) Calcule $\iint_{\Omega} \frac{1}{x-y} \cdot \cos(2x+3y)\,dx\,dy$, sendo Ω o quadrilátero com vértices nos pontos $A\left(\frac{3}{5}, -\frac{2}{5}\right)$; $B\left(\frac{6}{5}, -\frac{4}{5}\right)$; $C\left(\frac{9}{5}, \frac{-1}{5}\right)$ e $D\left(\frac{6}{5}, \frac{1}{5}\right)$.

Solução: Escreva: $\begin{cases} u = x - y \\ v = 2x + 3y \end{cases}$ $\longrightarrow$ $x = u + y$

$$v = 2.(u+y) + 3y$$

$$v = 2u + 5y$$

$$\boxed{y = -\frac{2}{5}u + \frac{1}{5}v}$$

Disso, tem-se:

$$x = u + y$$

$$x = u - \frac{2}{5}u + \frac{1}{5}v$$

$$\boxed{x = \frac{3}{5}u + \frac{1}{5}v}$$

Assim, segue que:

$$J(T)(u,v) = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ \frac{-2}{5} & \frac{1}{5} \end{bmatrix}$$

$$\Rightarrow \det J(T)(u,v) = \begin{vmatrix} \frac{3}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{vmatrix} = \frac{3}{5}\cdot\frac{1}{5} - \frac{1}{5}\cdot\left(-\frac{2}{5}\right)$$

$$= \frac{3}{25} + \frac{2}{25} = \frac{1}{5}$$

Assim, temos:

$$\iint_{\Omega} \frac{1}{\underbrace{x-y}_{u}} \cdot \cos(\underbrace{2x+3y}_{v})\,dx\,dy = \iint_{\Omega'} \frac{1}{u} \cdot \cos v \cdot \underbrace{|\det J(T)(u,v)|}_{\frac{1}{5}}\,du\,dv$$

$$= \iint_{\Omega'} \frac{1}{u} \cdot \cos v \cdot \left|\frac{1}{5}\right| \cdot du\,dv = \frac{1}{5}\iint_{\Omega'} \frac{1}{u}\cdot \cos v\,du\,dv$$

Resta determinar a região Ω' a partir de Ω e de T.

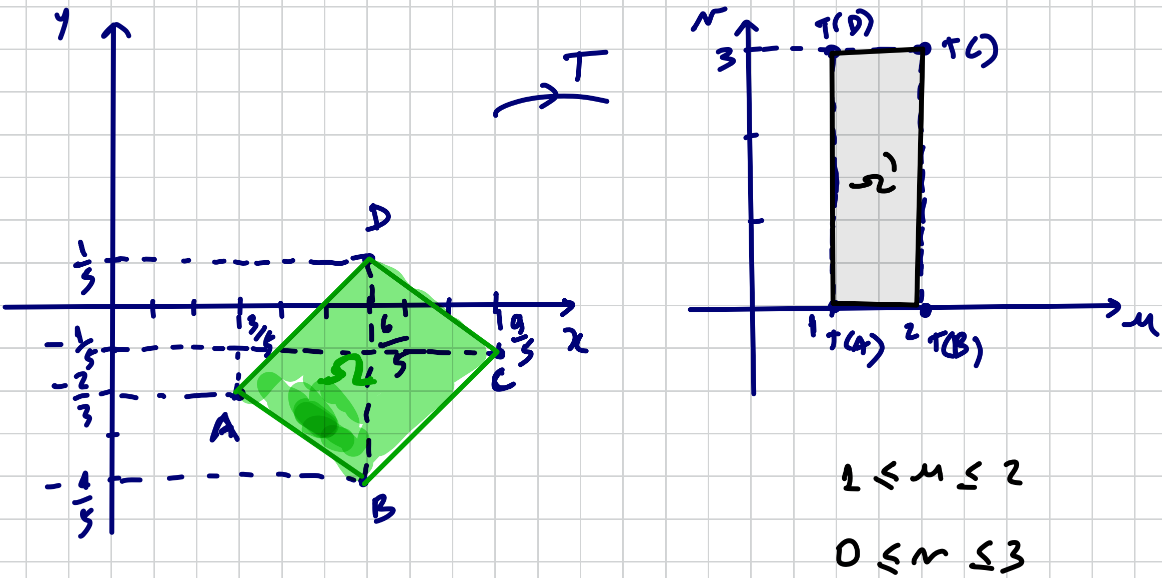
$$(x,y) \xmapsto{T} (u,v) = (x-y,\, 2x+3y)$$

$$A\left(\frac{3}{5}, -\frac{2}{5}\right) \xmapsto{T} T(A) = \left(\frac{3}{5} - \left(-\frac{2}{5}\right), \frac{6}{5} - \frac{6}{5}\right) = (1,0)$$

$$B\left(\frac{6}{5}, -\frac{4}{5}\right) \xmapsto{T} T(B) = \left(\frac{6}{5} + \frac{4}{5}, \frac{12}{5} - \frac{12}{5}\right) = (2,0)$$

$$C\left(\frac{9}{5}, -\frac{1}{5}\right) \xmapsto{T} T(C) = \left(\frac{9}{5} + \frac{1}{5}, \frac{18}{5} - \frac{3}{5}\right) = (2,3)$$

$$D\left(\frac{6}{5}, \frac{1}{5}\right) \xmapsto{T} T(D) = \left(\frac{6}{5} - \frac{1}{5}, \frac{12}{5} + \frac{3}{5}\right) = (1, 3)$$



$1 \leq u \leq 2$

$0 \leq v \leq 3$

Assim, obtemos:

$$\iint_{\Omega} \frac{1}{x-y} \cdot \cos(2x+3y)\, dx\, dy = \frac{1}{5} \iint_{\Omega'} \frac{1}{u} \cos v \, du\, dv$$

$$= \frac{1}{5} \int_{v=0}^{v=3} \int_{u=1}^{u=2} \frac{1}{u} \cdot \cos v \, du\, dv =$$

$$= \frac{1}{5} \cdot \int_{v=0}^{v=3} \cos v \, dv \cdot \int_{u=1}^{u=2} \frac{du}{u} = \frac{1}{5} \cdot \operatorname{sen} v \Big|_{v=0}^{v=3} \cdot \ln|u| \Big|_{u=1}^{u=2} =$$

$$= \frac{1}{5} \cdot (\operatorname{sen} 3 - \underbrace{\operatorname{sen} 0}_{=0}) \cdot (\ln 2 - \underbrace{\ln 1}_{=0}) =$$

$$= \frac{1}{5} \cdot \operatorname{sen} 3 \cdot \ln 2$$