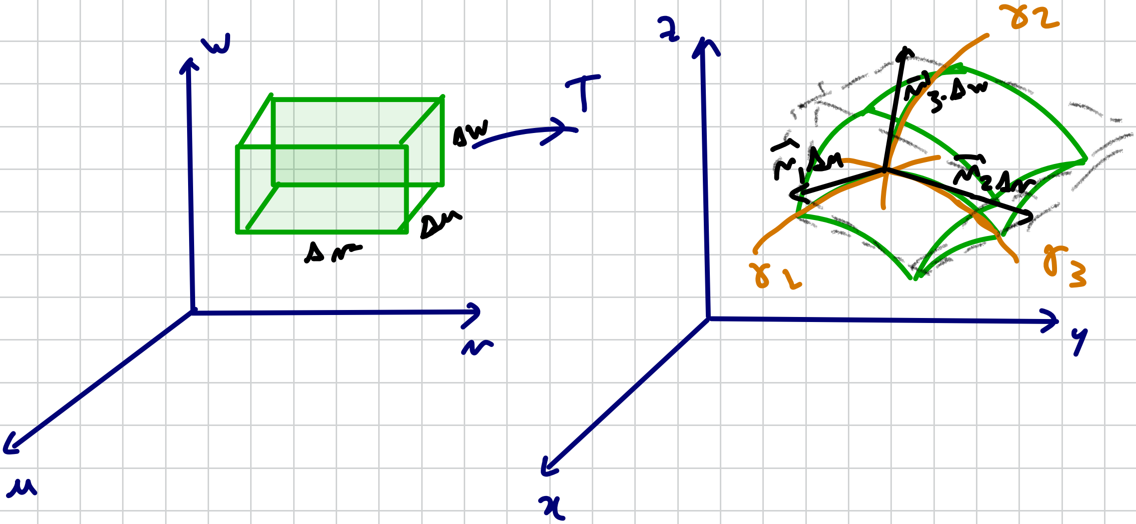
CÁLCULO 3, TURMA T3:

$\boxed{\text{29/06/26 – AULA 37}}$

Na aula passada iniciamos o estudo sobre mudança de variáveis no $\mathbb{R}^3$.



$$\vec{r}_1 \Delta u = \left(\frac{\partial x}{\partial u}\Delta u, \frac{\partial y}{\partial u}\Delta u, \frac{\partial z}{\partial u}\Delta u\right)(u_0,v_0,w_0)$$

$$\vec{r}_2 \Delta v = \left(\frac{\partial x}{\partial v}\Delta v, \frac{\partial y}{\partial v}\Delta v, \frac{\partial z}{\partial v}\Delta v\right)(u_0,v_0,w_0)$$

$$\vec{r}_3 \cdot \Delta w = \left(\frac{\partial x}{\partial w}\cdot\Delta w, \frac{\partial y}{\partial w}\Delta w, \frac{\partial z}{\partial w}\Delta w\right)(u_0,v_0,w_0)$$

O volume do "paralelepípedo oblíquo"

é aproximado pelo volume do paralelepípedo formado pelos vetores $\vec{v}_1\,\Delta u$, $\vec{v}_2\,\Delta v$ e $\vec{v}_3\,\Delta w$, como estudado na ALGA; ou seja será o produto misto entre estes três vetores:

$$V \cong \left|\,[\vec{v}_1\,\Delta u,\ \vec{v}_2\,\Delta v,\ \vec{v}_3\,\Delta w]\,\right| =$$

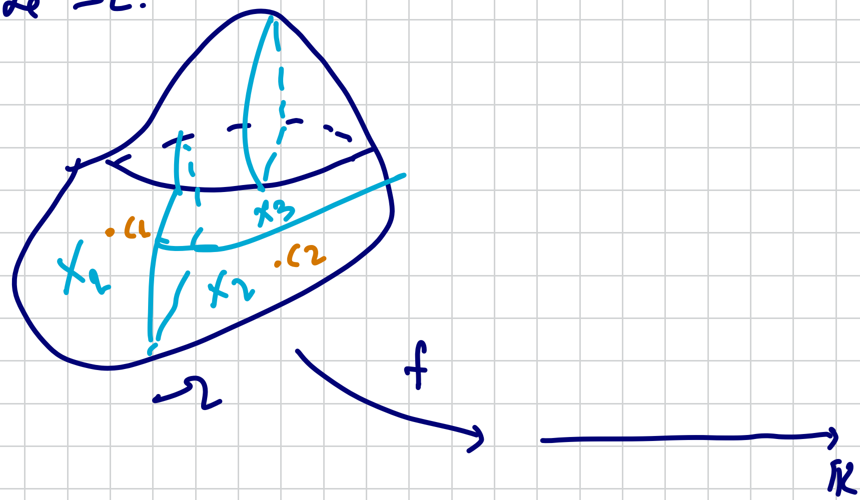
$$= \left| \begin{vmatrix} \frac{\partial x}{\partial u}\,\Delta u & \frac{\partial y}{\partial u}\,\Delta u & \frac{\partial z}{\partial u}\,\Delta u \\ \frac{\partial x}{\partial v}\,\Delta v & \frac{\partial y}{\partial v}\,\Delta v & \frac{\partial z}{\partial v}\,\Delta v \\ \frac{\partial x}{\partial w}\,\Delta w & \frac{\partial y}{\partial w}\,\Delta w & \frac{\partial z}{\partial w}\,\Delta w \end{vmatrix} \right| =$$

$$= \left| \underbrace{\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix}}_{\det J(T)(u,v,w)} \cdot \underbrace{\Delta u}_{>0} \cdot \underbrace{\Delta v}_{>0} \cdot \underbrace{\Delta w}_{>0} \right|$$

$$= \left|\det J(T)(u,v,w)\right| \cdot \Delta u \cdot \Delta v \cdot \Delta w$$

Seja $f : \Omega \subset \mathbb{R}^3 \to \mathbb{R}$ uma função integrável.

Seja $D = \{X_1, X_2, \dots, X_n\}$ uma decomposição de Ω.



Montamos a seguinte soma de Riemann:

$$\Sigma(f; D) = \sum_{i=1}^{n} f(c_i) \cdot \mathrm{Vol}(X_i),$$

onde c_i é um ponto qualquer em cada X_i.

Assim, temos:

$\mathrm{Vol}(X_i) \approx V_i$ de cada paralelepípedo oblíquo.

$$\mathrm{Vol}(X_i) \approx |\det J(T)(u_i, v_i, w_i)| \cdot \Delta u_i \, \Delta v_i \, \Delta w_i$$

Disso, segue que:

$$\Sigma(f;D) \approx \sum_{i=1}^{n} f(c_i) . |\det J(T)(u_i,v_i,w_i)| \, \Delta u_i \, \Delta v_i \, \Delta w_i$$

Tomando o limite com $n \to \infty$, obtemos:

$$\underbrace{\iiint_{\Omega} f(x,y,z)\, dV}_{} = \lim_{n\to\infty} \Sigma(f;D) =$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i) \, |\det J(T)(u_i,v_i,w_i)| . \Delta u_i . \Delta v_i . \Delta w_i$$

$$\underbrace{= \iiint_{\Omega'} f(x(u,v,w), y(u,v,w), z(u,v,w)) . |\det J(T)(u,v,w)| . du\,dv\,dw}_{}$$

FÓRMULA DA MUDANÇA DE COORDENADAS NO $\mathbb{R}^3$.

No que segue, apresentamos exemplos de aplicação:

01) Calcule $\iiint_\Omega \frac{e^{x-y+z}}{x+y-z}\,dx\,dy\,dz$, onde Ω é a região do $\mathbb{R}^3$ dada por:

$$0 \leq x-y+z \leq 2$$

$$1 \leq x+y-z \leq 2$$

$$0 \leq z \leq 1.$$

<u>Solução:</u> Escreva

$$T: \begin{cases} u = x-y+z \\ v = x+y-z \\ w = z \end{cases} \longrightarrow \begin{array}{l} 0 \leq u \leq 2 \\ 1 \leq v \leq 2 \\ 0 \leq w \leq 1 \end{array}$$

$$\begin{array}{r} u = x - \cancel{y} + \cancel{z} \\ + \quad v = x + \cancel{y} - \cancel{z} \\ \hline u + v = 2x \end{array} \Rightarrow \boxed{x = \frac{1}{2}u + \frac{1}{2}v}$$

$$\boxed{z = w}$$

$$v = x+y-z$$

$$\Rightarrow y = v - x + z$$

$$y = v - \frac{1}{2}u - \frac{1}{2}v + w$$

$$\boxed{y = -\frac{1}{2}u + \frac{1}{2}v + w}$$

Assim, a matriz Jacobiana será:

$$
J(T)(u,v,w) = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{bmatrix}
$$

$$
= \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 1 \end{bmatrix}
$$

$$
\Rightarrow \det J(T)(u,v,w) = \left| \begin{array}{ccc|cc} \frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 1 & 0 & 1 \end{array} \right.
$$

$$
= \frac{1}{4} + 0 + 0 - 0 - 0 + \frac{1}{4} = \frac{1}{2}
$$

Assim, vamos obter:

$$\iiint_{\Omega} \frac{e^{x-y+z}}{x+y-z}\, dx\,dy\,dz =$$

$$= \iiint_{\Omega'} \frac{e^{u}}{v} \cdot \underbrace{|\det J(T)(u,v,w)|}_{\frac{1}{2}} \cdot du\,dv\,dw =$$

$$= \frac{1}{2} \cdot \int_{w=0}^{w=1} \int_{v=1}^{v=2} \int_{u=0}^{u=2} \frac{e^{u}}{v} \cdot du\,dv\,dw =$$

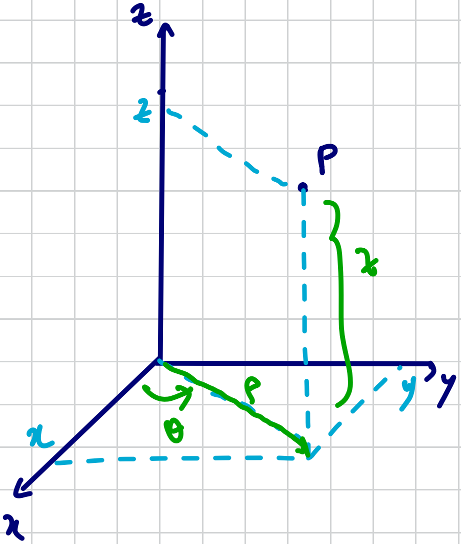
$$= \frac{1}{2} \cdot \int_{w=0}^{w=1} dw \cdot \int_{v=1}^{v=2} \frac{dv}{v} \cdot \int_{u=0}^{u=2} e^{u}\, du =$$

$$= \frac{1}{2} \cdot w\Big|_{0}^{1} \cdot \ln|v|\Big|_{v=1}^{v=2} \cdot e^{u}\Big|_{u=0}^{u=2} =$$

$$= \frac{1}{2} \cdot (1-0) \cdot \Big(\ln 2 - \underbrace{\ln 1}_{=0}\Big) \cdot \Big(e^{2} - \underbrace{e^{0}}_{=1}\Big)$$

$$\ln 2 \cdot \frac{(e^{2}-1)}{2} .$$

02) SISTEMA DE COORDENADAS CILÍNDRICAS: É o sistema de coordenadas polares para o $\mathbb{R}^3$.



$$P(x,y,z) \leftrightarrow P(\rho,\theta,z)$$

onde:

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\,\text{sen}\,\theta. \end{cases}$$

Neste caso, dada $f: \Omega \subset \mathbb{R}^3 \to \mathbb{R}$ integrável, então:

$$T: \begin{cases} x = \rho\cos\theta \\ y = \rho\,\text{sen}\,\theta \\ z = z \end{cases}$$

Disso:

$$J(T)(\rho,\theta,z) = \begin{bmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial y}{\partial \rho} & \frac{\partial z}{\partial \rho} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial z} & \frac{\partial y}{\partial z} & \frac{\partial z}{\partial z} \end{bmatrix} =$$

$$= \begin{bmatrix} \cos\theta & \operatorname{sen}\theta & 0 \\ -\rho\operatorname{sen}\theta & \rho\cdot\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Então:

$$\det J(T)(\rho,\theta,z) = \begin{vmatrix} \cos\theta & \operatorname{sen}\theta & 0 \\ -\rho\operatorname{sen}\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{matrix} \cos\theta & \operatorname{sen}\theta \\ -\rho\operatorname{sen}\theta & \rho\cos\theta \\ 0 & 0 \end{matrix}$$

$$= \rho\cos^2\theta + 0 + 0 - 0 - 0 + \rho\operatorname{sen}^2\theta$$

$$= \rho\cos^2\theta + \rho\operatorname{sen}^2\theta = \rho\underbrace{(\cos^2\theta + \operatorname{sen}^2\theta)}_{=1} = \rho$$

Assim, obtemos:

$$\iiint_{\Omega} f(x,y,z)\,dx\,dy\,dz =$$

$$= \iiint_{\Omega'} f(x(\rho,\theta,z), y(\rho,\theta,z), z(\rho,\theta,z))\,\underline{\rho}\,.\,d\rho\,d\theta\,.\,dz$$

$\hookrightarrow |\det J(T)(\rho,\theta,z)|$