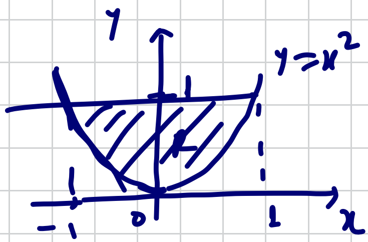
CÁLCULO 3, TURMA T3.

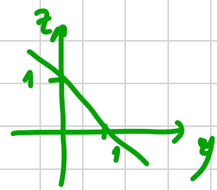
26/06/26 – AULA 36

Na aula passada iniciamos integrais triplas.

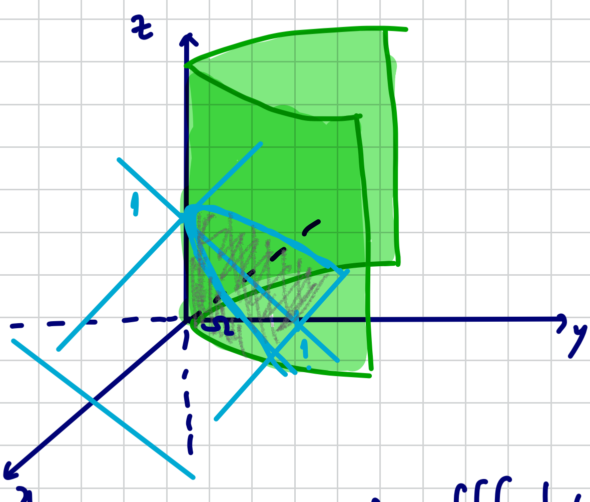
Ficou um exercício pendente:

02) Use integral tripla para determinar o volume do sólido limitado pelo cilindro $y = x^2$ e pelos planos $z = 0$ e $y + z = 1$. (Resp.: $\frac{8}{15}$ u.v.)

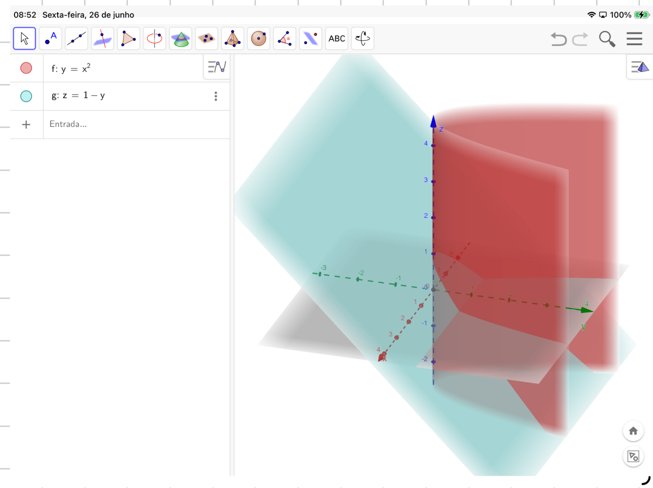




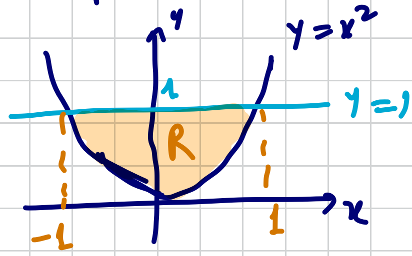
<u>SOLUÇÃO:</u>



$$V = \iiint_{\Omega} dV = \iint_R dx\,dy \int_{z=0}^{z=1-y} dz =$$



No plano xy temos:



$$= \int_{x=-1}^{x=1}\int_{y=x^2}^{y=1} dx\,dy \int_{z=0}^{z=1-y} dz = 2\cdot\int_{x=0}^{x=1}\int_{y=x^2}^{y=1} dx\,dy\int_{z=0}^{z=1-y} dz =$$

DEVIDO À SIMETRIA DO SÓLIDO S NO $\mathbb{R}^3$

$$= 2\cdot\int_{x=0}^{x=1}\int_{y=x^2}^{y=1} dx\,dy \;\; z\Big|_{z=0}^{z=1-y} = 2\cdot\int_{x=0}^{x=1}\int_{y=x^2}^{y=1}(1-y)\,dy\,dx =$$

$$= 2\cdot\int_{x=0}^{x=1}\left(y-\frac{y^2}{2}\right)\Bigg|_{y=x^2}^{y=1} dx = 2\cdot\int_{x=0}^{x=1}\left[1-\frac{1}{2}-\left(x^2-\frac{x^4}{2}\right)\right]dx$$

$$= 2 \cdot \int_0^1 \left(\frac{1}{2} - x^2 + \frac{x^4}{2}\right) dx = 2 \cdot \left[\frac{x}{2} - \frac{x^3}{3} + \frac{1}{2} \cdot \frac{x^5}{5}\right]\Bigg|_0^1 =$$

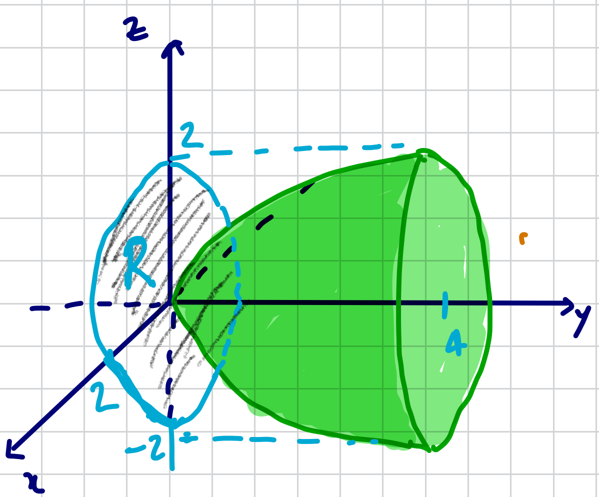
$$= 2\left(\frac{1}{2} - \frac{1}{3} + \frac{1}{10} - 0\right) = 2 \cdot \left(\frac{15 - 10 + 3}{30}\right) =$$

$$= \cancel{2} \cdot \frac{8}{\cancel{30}_{15}} = \frac{8}{15} \text{ u.v}$$

unidades de volume.

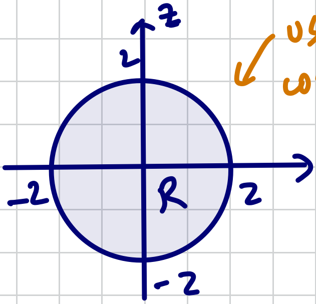
03) Calcule $\iiint_\Omega \sqrt{x^2 + z^2}\, dV$, onde Ω é a região limitada pelo paraboloide $y = x^2 + z^2$ e pelo plano $y = 4$.

Solução:



$y = 4$: no plano xz temos:

$x^2 + z^2 = 4$ (circunf.)



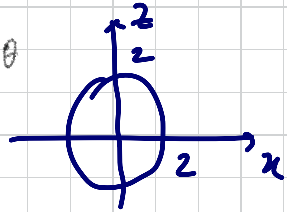
USAREMOS COORD. POLARES NO PLANO xz

Assim, teremos:

$$\iiint_{\Omega} \sqrt{x^2+z^2}\, dV = \iint_R dx\,dz \int_{y=x^2+z^2}^{y=4} \underbrace{\sqrt{x^2+z^2}}_{\text{constante para } y}\, dy =$$

$$= \iint_R \sqrt{x^2+z^2}\, dx\,dz \int_{y=x^2+z^2}^{y=4} dy = \iint_R \sqrt{x^2+z^2}\cdot y \Big|_{y=x^2+z^2}^{y=4} dx\,dz =$$

$$= \iint_R \underbrace{\sqrt{x^2+z^2}}_{\rho}\,\left(4-\underbrace{(x^2+z^2)}_{\rho^2}\right) \underbrace{dx\,dz}_{\rho\, d\rho\, d\theta} =$$



$$= \int_{\theta=0}^{\theta=2\pi} \int_{\rho=0}^{\rho=2} \rho\cdot(4-\rho^2)\cdot\rho\, d\rho\, d\theta =$$

$$\boxed{\rho^2 = x^2+z^2}$$

$$= \int_{\theta=0}^{\theta=2\pi} d\theta \cdot \int_{\rho=0}^{\rho=2} (4\rho^2 - \rho^4)\, d\rho = \theta \Big|_{\theta=0}^{\theta=2\pi} \cdot \left(\frac{4\rho^3}{3} - \frac{\rho^5}{5} \right) \Big|_{\rho=0}^{\rho=2}$$

$$= 2\pi \cdot \left(\frac{32}{3} - \frac{32}{5} \right) = 2\pi \cdot \frac{160 - 96}{15} =$$

$$= 2\pi \cdot \frac{64}{15} = \frac{128\pi}{15} \text{ u.v.}$$

MUDANÇA DE VARIÁVEIS NO $\mathbb{R}^3$

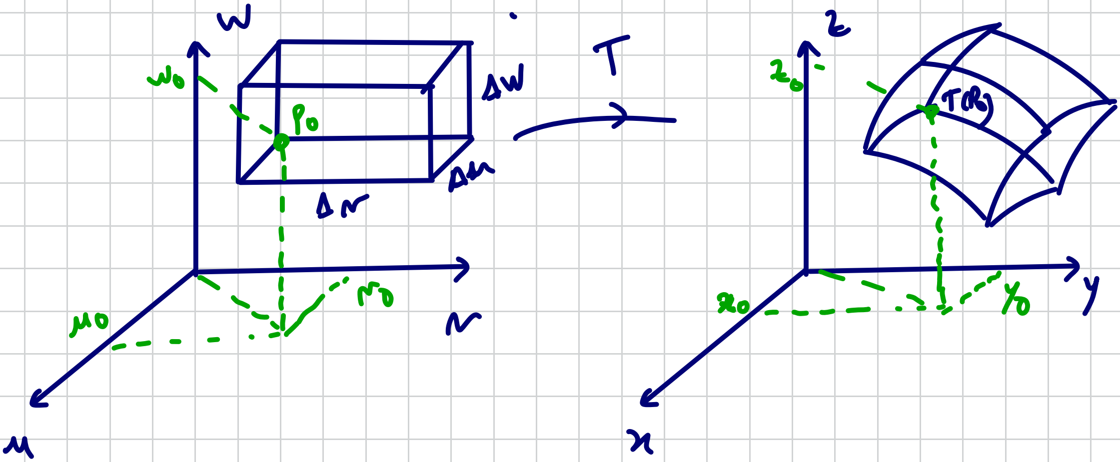
Do mesmo modo que trabalhamos no $\mathbb{R}^2$, existe o teorema para mudar de variáveis no $\mathbb{R}^3$, exatamente a mesma fórmula, à qual apresentamos a seguir.

Seja $f : \Omega \subset \mathbb{R}^3 \to \mathbb{R}$ uma função integrável em Ω. Seja $T : \Omega \subset \mathbb{R}^3 \to T(\Omega) = \Omega'$ uma transformação injetiva e de classe C^1 (com derivadas parciais contínuas)

$T(x,y,z) = (u, v, w)$, onde

$$\begin{cases} u = u(x,y,z) \\ v = v(x,y,z) \\ w = w(x,y,z) \end{cases}$$

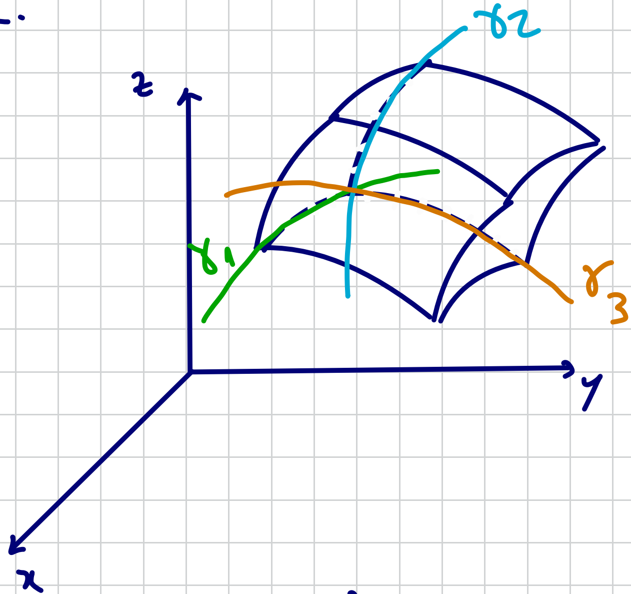
Seja A um paralelepípedo (bloco) no sistema u, v, w, de lados Δu, Δv e Δw.



Seja $P_0(u_0, v_0, w_0)$ um vértice conforme o esquema acima.

Então $T(P_0) = (x_0, y_0, z_0)$ no sistema xyz.

Considere as curvas γ_1, γ_2 e γ_3 no sistema xyz, passando pelo ponto $T(P_0)$, arestas la figure.



$$\gamma_1 : [u_0, u_0+\Delta u] \to \mathbb{R}^3$$

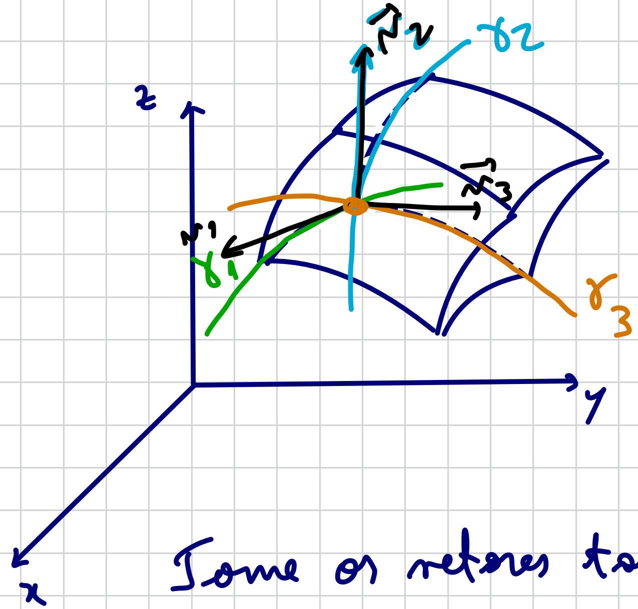
$$\gamma_1(u) = \big(x(u, v_0, w_0), y(u, v_0, w_0), z(u, v_0, w_0)\big)$$

$$\gamma_2 : [v_0, v_0+\Delta v] \to \mathbb{R}^3$$

$$\gamma_2(v) = \big(x(u_0, v, w_0), y(u_0, v, w_0), z(u_0, v, w_0)\big)$$

$$\gamma_3 : [w_0, w_0+\Delta w] \to \mathbb{R}^3$$

$$\gamma_3(w) = \big(x(u_0, v_0, w), y(u_0, v_0, w), z(u_0, v_0, w)\big)$$



Tome os vetores tangentes $\vec{v}_1$, $\vec{v}_2$ e $\vec{v}_3$, no ponto $T(P_0)$, dados por:

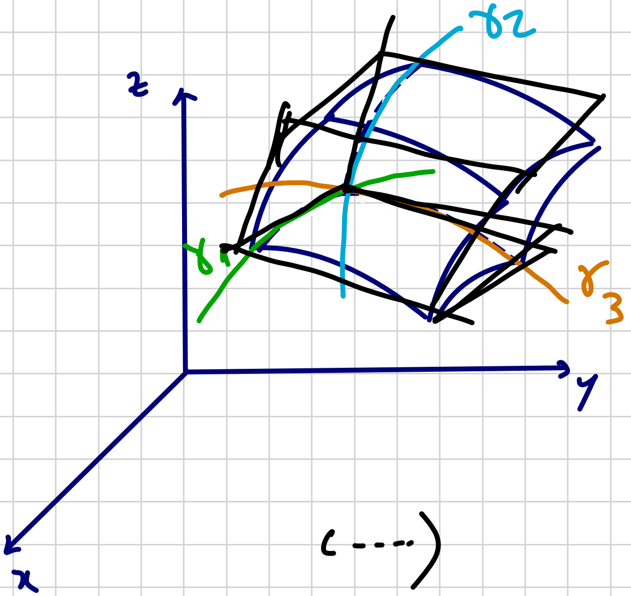
$$\vec{v}_1 = \gamma_1'(u_0) = \left(\frac{\partial x}{\partial u}(u_0,v_0,w_0), \frac{\partial y}{\partial u}(u_0,v_0,w_0), \frac{\partial z}{\partial u}(u_0,v_0,w_0)\right)$$

$$\vec{v}_2 = \gamma_2'(v_0) = \left(\frac{\partial x}{\partial v}(u_0,v_0,w_0), \frac{\partial y}{\partial v}(u_0,v_0,w_0), \frac{\partial z}{\partial v}(u_0,v_0,w_0)\right)$$

$$\vec{v}_3 = \gamma_3'(w_0) = \left(\frac{\partial x}{\partial w}(u_0,v_0,w_0), \frac{\partial y}{\partial w}(u_0,v_0,w_0), \frac{\partial z}{\partial w}(u_0,v_0,w_0)\right)$$

Tome os vetores $\vec{v}_1 \Delta u$, $\vec{v}_2 \Delta v$ e $\vec{v}_3 \Delta w$.

O volume do bloco "deformado" será aproximadamente o volume do paralelepípelo dado por $\vec{r}_1' \Delta u$, $\vec{r}_2' \Delta v$ e $\vec{r}_3' \Delta w$:



(.....)

CONTINUAREMOS A CONSTRUÇÃO NA PRÓXIMA AULA.