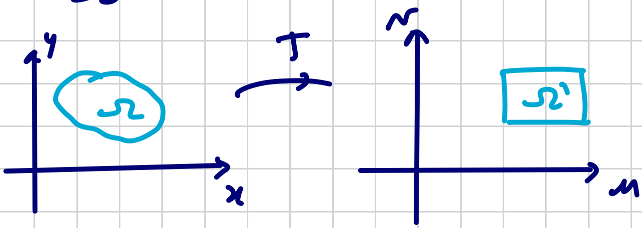
CÁLCULO 3, TURMA T3

19/06/26 – AULA 33

Na aula passada a mudança geral de variáveis em integrais duplas:

$$\iint_{\Omega} f(x,y)\,dA = \iint_{\Omega'} f(x(u,v), y(u,v))\,|\det J(T)(u,v)|\,du\,dv$$



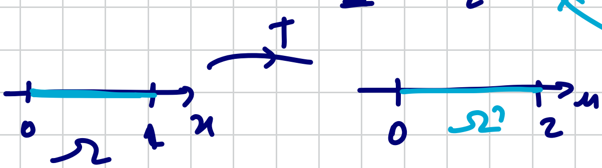
Na verdade isso já era passado no cálculo 2:

$$\int_{x=0}^{x=1} \text{sen}\,2x\,dx = \int_{u=0}^{u=2} \text{sen}\,u \cdot \frac{1}{2}\,du = \frac{1}{2}\int_0^2 \text{sen}\,u\,du =$$

$T: \ u = 2x \rightarrow x = \frac{1}{2}u$

$du = J(T)(u) = \frac{1}{2} \Longrightarrow \det J(T)(u) = \frac{1}{2}$

o determinante de um número real é o próprio número real.



$x = 0 \Rightarrow u = 0$

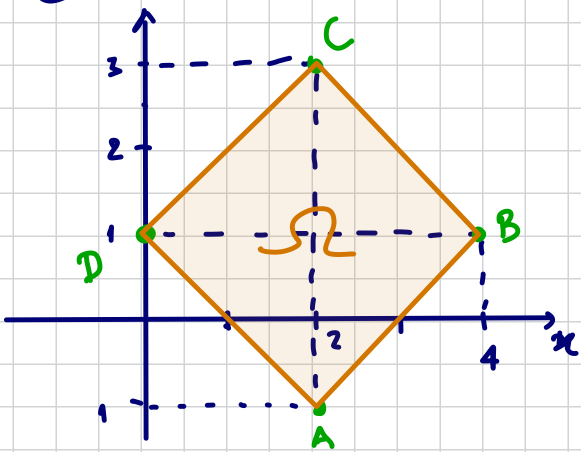
$x = 1 \Rightarrow u = 2$

$$= \frac{1}{2}(-\cos u)\Big|_0^2 = -\frac{1}{2}(\cos 2 - \cos 0) = -\frac{1}{2}(\cos 2 - 1)$$

Vejamos outros exemplos interessantes:

02) Calcule $\iint_{\Omega} (x-y)^2 \ln(x+y)\,dx\,dy$, onde Ω é o quadrilátero com vértices nos pontos $A(2,-1)$; $B(4,1)$; $C(2,3)$ e $D(0,1)$.

SOLUÇÃO:



A região Ω é um pouco trabalhosa. Mas o pior é não poder "separar" as variáveis x e y da integral.

Por isso, vamos mudar de variável.

$$\iint_{\Omega} (\underbrace{x-y}_{u})^2 \cdot \ln(\underbrace{x+y}_{v})\,dx\,dy$$

Como $\begin{cases} u = x - y \\ v = x + y \end{cases}$ (mas precisamos de x e y em termos de u e v)

$$+\left\{\begin{aligned} u &= x-y \\ v &= x+y \end{aligned}\right.$$

$$2x = u+v \Rightarrow \boxed{x = \frac{1}{2}u + \frac{1}{2}v}$$

$$v = x+y \Rightarrow y = v - x$$

$$y = v - \left(\frac{1}{2}u + \frac{1}{2}v\right)$$

$$\boxed{y = -\frac{1}{2}u + \frac{1}{2}v}$$

$$T(u,v) = (x,y) = \Big(\overbrace{\frac{1}{2}u + \frac{1}{2}v}^{x}, \overbrace{-\frac{1}{2}u + \frac{1}{2}v}^{y}\Big)$$

$$J(T)(u,v) = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\Rightarrow \det J(T)(u,v) = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{4} - \left(-\frac{1}{4}\right) = \frac{1}{2}$$

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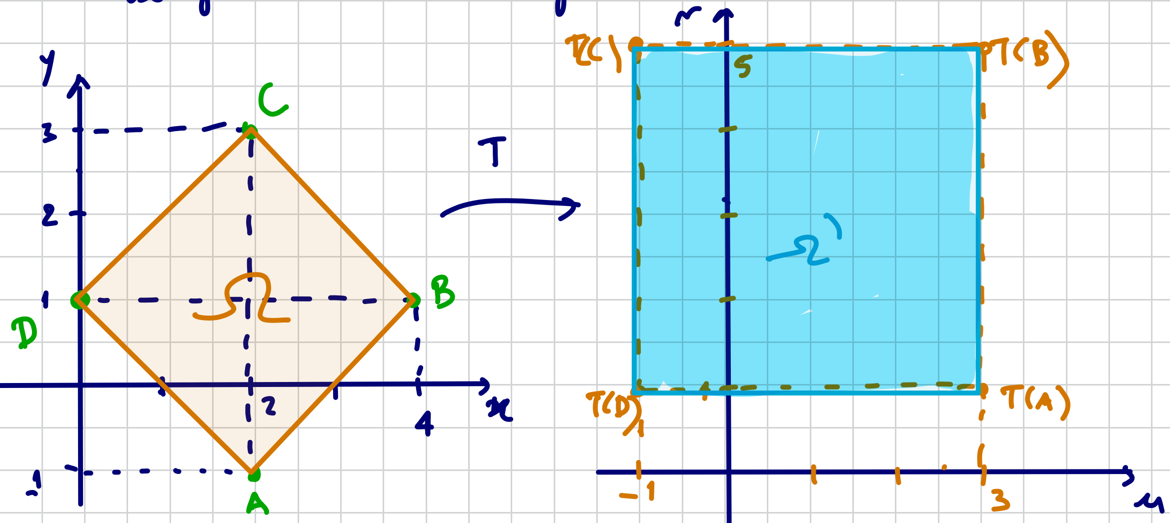
Disso, temos:

$$\iint_{\Omega} \underbrace{(x-y)}_{u}{}^2 \cdot \ln\underbrace{(x+y)}_{v}\,dx\,dy = \iint_{\Omega'} u^2 \cdot \ln v \,\underbrace{|\det J(T)(u,v)|}_{\frac{1}{2}} \cdot du\,dv$$

$$= \iint_{\Omega'} u^2 \cdot \ln v \cdot \frac{1}{2} \cdot du\,dv =$$

$$= \frac{1}{2} \iint_{\Omega'} u^2 \ln v \, du\,dv.$$

Só falta obter a região Ω'.



$$(x,y) \xrightarrow{\;T\;} (u = x-y,\ v = x+y)$$

$$A(2,-1) \longmapsto T(A) = (2-(-1),\ 2-1) = (3,1)$$

$$B(4,1) \longmapsto T(B) = (4-1,\ 4+1) = (3,5)$$

$$C(2,3) \longmapsto T(C) = (2-3,\ 2+3) = (-1,5)$$

$$D(0,1) \longmapsto T(D) = (0-1,\ 0+1) = (-1,1)$$

$$\Rightarrow \quad -1 \leq u \leq 3 \quad e \quad 1 \leq v \leq 5$$

Logo:

$$\iint_{\Omega} (x-y)^2 \ln(x+y)\,dx\,dy = \frac{1}{2}\iint_{\Omega'} u^2 \ln v\,du\,dv =$$

$$= \frac{1}{2}\int_{v=1}^{v=5}\int_{u=-1}^{u=3} u^2 \ln v\,du\,dv = \frac{1}{2}\underbrace{\int_{v=1}^{v=5} \ln v\,dv}_{\text{integrar por partes.}} \cdot \int_{u=-1}^{u=3} u^2\,du =$$

$$\int \ln v\,dv = \int u\,.dw = u\cdot w - \int w\,du =$$

$$\begin{cases} u = \ln v \Rightarrow du = \dfrac{dv}{v} \\ dw = dv \Rightarrow w = v \end{cases}$$

$$= \ln v\,.\,v - \int \cancel{v}\,.\frac{dv}{\cancel{v}} = v\,.\ln v - v + C$$

$$= \frac{1}{2}\cdot\left(v\ln v - v\right)\Big|_1^5 \cdot \frac{u^3}{3}\Big|_{-1}^{3} =$$

$$= \frac{1}{2}\cdot\left(5\ln 5 - 5 - (\underbrace{1.\ln 1}_{0} - 1)\right)\cdot\left(\frac{(3)^3}{3} - \frac{(-1)^3}{3}\right)$$

$$= \frac{1}{2}\left(5\ln 5 - 4\right)\cdot\left(\frac{28}{3}\right) = \frac{14}{3}\left(5\ln 5 - 4\right)$$

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03) Calcule $\iint_{\Omega} \frac{1}{x-y}\cdot\cos(2x+3y)\,dx\,dy$; onde Ω é o quadrilátero com vértices nos pontos $A\left(\frac{3}{5}, -\frac{2}{5}\right)$, $B\left(\frac{6}{5}; -\frac{4}{5}\right)$; $C\left(\frac{9}{5}, -\frac{1}{5}\right)$; $D\left(\frac{6}{5}; \frac{1}{5}\right)$.

Solução: Escreva $\begin{cases} u = x-y \longrightarrow x = u+y \\ v = 2x+3y \end{cases}$

$$v = 2(u+y) + 3y$$

$$v = 2u + 5y$$

$$\Rightarrow \boxed{y = -\frac{2}{5}u + \frac{1}{5}v}$$

$$\Rightarrow x = u + y$$

$$x = u + \left(-\frac{2}{5}u + \frac{1}{5}v\right)$$

$$\boxed{x = \frac{3}{5}u + \frac{1}{5}v}$$

$$T(u,v) = (x,y)$$

$$T(u,v) = \left(\frac{3}{5}u + \frac{1}{5}v,\ -\frac{2}{5}u + \frac{1}{5}v\right)$$

$$\underline{J(T)(u,v)} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix}$$

$$\Rightarrow \det J(T)(u,v) = \begin{vmatrix} \frac{3}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{vmatrix} =$$

$-$ $+$

$$= \frac{3}{5}\cdot\frac{1}{5} - \left(\frac{1}{5}\right)\cdot\left(-\frac{2}{5}\right) = \frac{3}{25} + \frac{2}{25} = \frac{1}{5}$$

$$\iint_{\Omega} \frac{1}{x-y}\cos(2x+3y)\,dx\,dy = \iint_{\Omega'} \frac{1}{u}\cdot\cos v\cdot \underbrace{|\det J(T)(u,v)|}_{\frac{1}{5}}\,du\,dv$$

$$= \iint_{\Omega'} \frac{1}{u}\cdot\cos v\cdot\frac{1}{5}\cdot du\,dv =$$

Vamos construir Ω' a partir de T e do Ω:

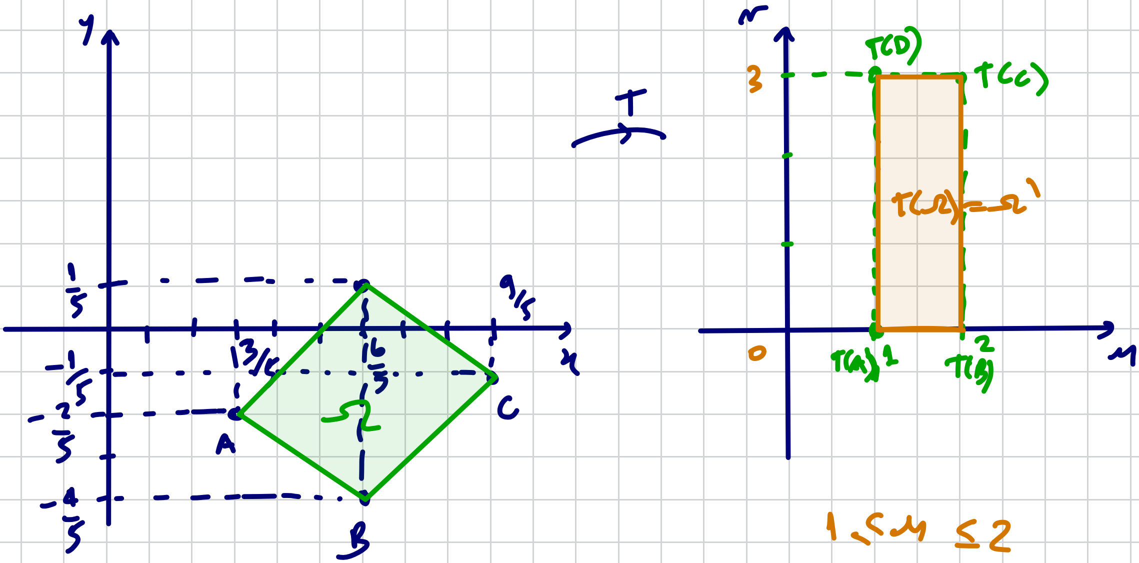
$$(x,y) \xrightarrow{\ T\ } (u,v) = (x-y,\ 2x+3y)$$

$$A\left(\frac{3}{5}, -\frac{2}{5}\right) \longmapsto T(A) = \left(\frac{3}{5}+\frac{2}{5}, \frac{6}{5}-\frac{6}{5}\right) = (1,0)$$

$$B\left(\frac{6}{5}, -\frac{4}{5}\right) \longmapsto T(B) = \left(\frac{6}{5}+\frac{4}{5}, \frac{12}{5}-\frac{12}{5}\right) = (2,0)$$

$$C\left(\frac{9}{5}, -\frac{1}{5}\right) \longmapsto T(C) = \left(\frac{9}{5}+\frac{1}{5}, \frac{18}{5}-\frac{3}{5}\right) = (2,3)$$

$$D\left(\frac{6}{5}, \frac{1}{5}\right) \longmapsto T(D) = \left(\frac{6}{5}-\frac{1}{5}, \frac{12}{5}+\frac{3}{5}\right) = (1,3)$$



$$1 \le u \le 2$$

$$0 \le v \le 3$$

Assim, obtemos:

$$\stackrel{\text{Fubini}}{=} \frac{1}{5}\int_{v=0}^{v=3}\int_{u=1}^{u=2} \frac{1}{u}\cdot \cos v \, du\, dv =$$

$$= \frac{1}{5}\cdot \int_{v=0}^{v=3} \cos v \, dv \cdot \int_{u=1}^{u=2} \frac{du}{u} =$$

$$\frac{1}{5}\cdot (\operatorname{sen} v)\Big|_0^3 \Big(\ln|u|\Big)\Big|_1^2 =$$

$$= \frac{1}{5}\cdot(\operatorname{sen}3 - \underbrace{\operatorname{sen}0}_{0''})\cdot(\ln 2 - \underbrace{\ln 1}_{0''})$$

$$= \frac{1}{5}\cdot \operatorname{sen}3 \cdot \ln 2 = \frac{1}{5}\ln 2 \cdot \operatorname{sen}3$$