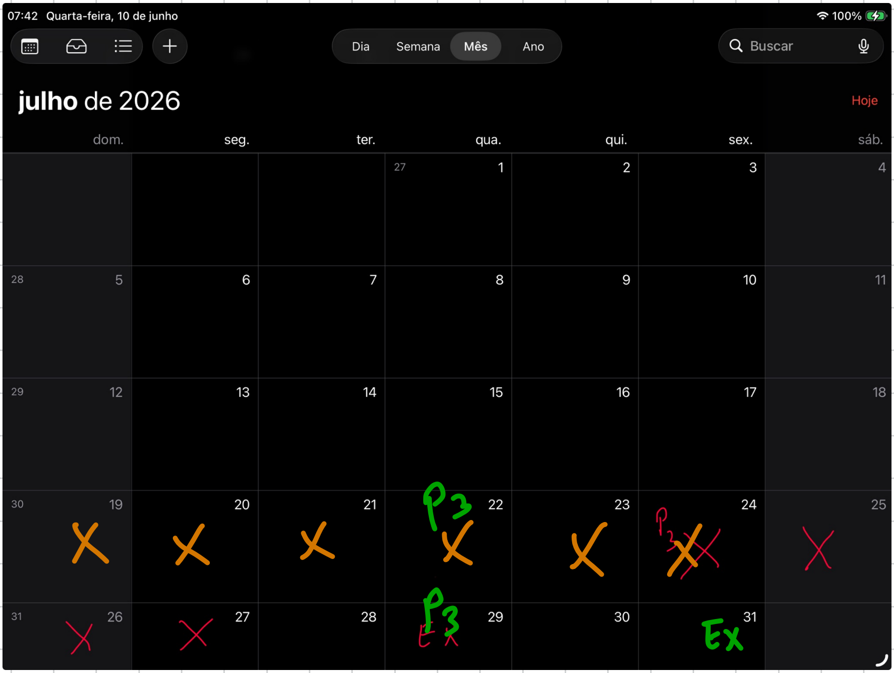
CÁLCULO 3, TURMA T3 10/06/26 – AULA 30.

Obs: NOVAS DATAS PARA P_3 e EXAME:



L5

2. De cada função vetorial a seguir, obtenha a matriz Jacobiana:

(a) $f(x,y) = (e^{x^2+y^2}, 2x^2y + 3y^2, \sqrt{x^2+y^2})$

(b) $f(x,y,z) = (x^2+y^2+z^2, xyz, \cos xy, x^2 - yz)$

(a) $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$

$$[df_a]_{3\times 2} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \\ f_{31} & f_{32} \end{bmatrix} \qquad \cdot \; f_{ij} = \frac{\partial f_i}{\partial x_j}$$

onde:

$$f_1(x,y) = e^{x^2+y^2}$$

$$\frac{\partial f_1}{\partial x} = 2x e^{x^2+y^2} \; ; \qquad \frac{\partial f_1}{\partial y} = 2y\, e^{x^2+y^2}$$

$$f_2(x,y) = 2x^2y + 3y^2$$

$$\frac{\partial f_2}{\partial x} = 4xy \qquad \frac{\partial f_2}{\partial y} = 2x^2 + 6y$$

$$f_3(x,y) = \left(x^2+y^2\right)^{1/2}$$

$$\frac{\partial f_3}{\partial x} = \frac{1}{\cancel{2}} \cdot \left(x^2+y^2\right)^{-1/2} \cdot \cancel{2}x = \frac{x}{\sqrt{x^2+y^2}} \qquad \frac{\partial f_3}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

Assim:

$$df_a = \begin{bmatrix} 2x \cdot e^{x^2+y^2} & 2y \cdot e^{x^2+y^2} \\ 4xy & 2x^2+6y \\ \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \end{bmatrix}$$

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3. Seja $f : \mathbb{R}^2 \to \mathbb{R}$ dada por

$$f(x,y) = \begin{cases} \dfrac{3x^2y^2}{x^4+y^4}, & \text{se} \quad (x,y) \neq (0,0) \\ 0, & \text{se} \quad (x,y) = (0,0) \end{cases}.$$

Mostre que $\dfrac{\partial}{\partial x} f(0,0)$ e $\dfrac{\partial}{\partial y} f(0,0)$ existem, mas que f não é diferenciável na origem.

$$\bullet \ \frac{\partial f}{\partial x}(0,0) = \lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h} =$$

$$= \lim_{h \to 0} \frac{f(h,0) - 0}{h} = \lim_{h \to 0} \frac{\overbrace{\dfrac{3 \cdot h^2 \cdot 0^2}{h^4 + 0^4}}^{=0}}{h} = 0$$

$$\Rightarrow \boxed{\frac{\partial f}{\partial x}(0,0) = 0}$$

$$\bullet \ \frac{\partial f}{\partial y}(0,0) = \lim_{h \to 0} \frac{f(0,0+h) - f(0,0)}{h} =$$

$$= \lim_{h \to 0} \frac{f(0,h) - 0}{h} = \frac{\left(\dfrac{3 \cdot 0^2 \cdot h^2}{0^4 + h^4}\right)^{=0}}{h} = 0$$

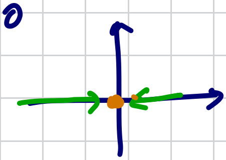
$$\Rightarrow \boxed{\frac{\partial f}{\partial y}(0,0) = 0}$$

Segunda parte do exercício: Recorde que; se uma função $f : \Omega \subset \mathbb{R}^m \to \mathbb{R}^n$ for diferenciável, então f é contínua. Assim, por transposição, se f não for contínua em um ponto, ela não será diferenciável ali:

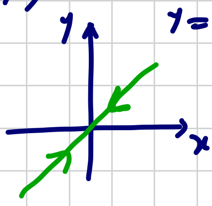
$$\left[\ P \Rightarrow Q \vdash \sim Q \Rightarrow \sim P\ \right]$$

Portanto, verifiquemos se f não é contínua na origem.

$$\bullet \lim_{\substack{(x,y)\to(0,0)\\ y=0}} f(x,y) = \lim_{\substack{(x,y)\to(0,0)\\ y=0}} \frac{3x^2y^2}{x^4+y^4} = \lim_{\substack{(x,y)\to(0,0)\\ y=0}} \frac{0}{x^4} = 0$$



$$\bullet \lim_{\substack{(x,y)\to(0,0)\\ y=x}} f(x,y) = \lim_{\substack{(x,y)\to(0,0)\\ y=x}} \frac{3x^2y^2}{x^4+y^4} = \lim_{\substack{(x,y)\to(0,0)\\ y=x}} \frac{3x^2\cdot x^2}{x^4+x^4} =$$



$$= \lim_{\substack{(x,y)\to(0,0)\\ y=x}} \frac{3\cancel{x^4}}{2\cancel{x^4}} = \frac{3}{2}$$

Logo, por caminhos diferentes obtemos limites diferentes, mostrando que f não é contínua na origem, e disso, não pode ser diferenciável na origem.

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5. Calcule as diferenciais totais de cada função a seguir:

(a) $z = \dfrac{x}{\sqrt{x^2+y^2}}$

(b) $z = \ln(xy+y^2)$

(c) $z = \arctan \dfrac{x+y}{1-xy}$

(d) $z = \dfrac{ye^x}{\sqrt{x^2+y^2}}$

(e) $z = \arcsin(x\sqrt{1-y^2} + y\sqrt{1-x^2})$

(f) $z = \dfrac{x \sin y}{\cos(xy)}$

(c) Lembre que o diferencial total de $w = f(x,y)$ é dado por:

$$df(x,y) = \frac{\partial f}{\partial x} \cdot dx + \frac{\partial f}{\partial y} dy$$

$$\Delta f = \underbrace{\frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y}_{df \text{ (DIFERENCIAL TOTAL)}} + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

No nosso caso:

$$f(x,y) = \arctan \frac{x+y}{1-xy}$$

$$(\arctan u)' = \frac{u'}{1+u^2}$$

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \text{ ; onde :}$$

$$\frac{\partial f}{\partial x} = \frac{\frac{\partial}{\partial x}\left(\frac{x+y}{1-xy}\right)}{1+\left(\frac{x+y}{1-xy}\right)^2} = \frac{\frac{(1-xy)\cdot 1-(x+y)\cdot(-y)}{(1-xy)^2}}{1+\frac{(x+y)^2}{(1-xy)^2}} =$$

$$= \frac{\frac{1-\cancel{xy}+\cancel{xy}+y^2}{\cancel{(1-xy)^2}}}{\frac{(1-xy)^2+x^2+2xy+y^2}{\cancel{(1-xy)^2}}} =$$

$$= \frac{1+y^2}{1-\cancel{2xy}+x^2y^2+x^2+\cancel{2xy}+y^2} = \frac{1+y^2}{\underbrace{x^2+x^2y^2}+y^2+1}$$

$$= \frac{1+y^2}{x^2(1+y^2)+(1+y^2)} = \frac{\cancel{1+y^2}}{\cancel{(1+y^2)}\left[x^2+1\right]} = \frac{1}{x^2+1}$$

$$\Rightarrow \frac{\partial f}{\partial x} = \frac{1}{x^2+1}$$

Analogamente; $\frac{\partial f}{\partial y} = \frac{1}{y^2+1}$. Dicho, obtenemos:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \frac{1}{x^2+1}dx + \frac{1}{y^2+1}dy.$$

L5

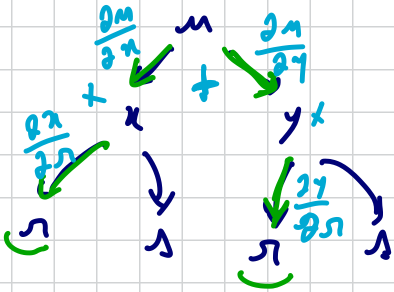
6. Calcule cada derivada parcial indicada, usando a regra da cadeia.

(a) $u = \ln xy + y^2$, onde $x = e^t$ e $y = e^{-t}$. Obter $\dfrac{\partial u}{\partial t}$.

(b) $u = x^2yz$, onde $x = \dfrac{r}{s}$, $y = re^s$ e $z = re^{-s}$. Obter $\dfrac{\partial u}{\partial r}$, $\dfrac{\partial u}{\partial s}$ e $\dfrac{\partial u}{\partial t}$.

(c) $u = \arcsin(3x + y)$, onde $x = r^2e^s$ e $y = \sin rs$. Obter $\dfrac{\partial u}{\partial r}$ e $\dfrac{\partial u}{\partial s}$.

(c) $$(\text{arcsen}\, u)' = \frac{u'}{\sqrt{1-u^2}}$$



$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}$$

onde:

- $$\frac{\partial u}{\partial x} = \frac{\frac{\partial}{\partial x}(3x+y)}{\sqrt{1-(3x+y)^2}} = \frac{3}{\sqrt{1-(3x+y)^2}}$$

- $$\frac{\partial u}{\partial y} = \frac{\frac{\partial}{\partial y}(3x+y)}{\sqrt{1-(3x+y)^2}} = \frac{1}{\sqrt{1-(3x+y)^2}}.$$

- $\dfrac{\partial x}{\partial r} = ?$ $\quad x = r^2 \cdot e^s \Rightarrow \dfrac{\partial x}{\partial r} = 2r \cdot e^s$

- $\dfrac{\partial y}{\partial r} = ?$ $\quad y = \text{sen}\, rs \Rightarrow \dfrac{\partial y}{\partial r} = s \cdot \cos(rs)$

$$\Rightarrow \frac{\partial u}{\partial r} = \frac{3}{\sqrt{1-(3x+y)^2}} \cdot 2r e^{s} + \frac{1}{\sqrt{1-(3x+y)^2}} \cdot s \cdot \cos(rs)$$

$$= \frac{1}{\sqrt{1-(3x+y)^2}} \cdot \left[6r e^{s} + s \cdot \cos(rs) \right]$$

$$= \frac{6r e^{s} + s \cdot \cos(rs)}{\sqrt{1-(3r^2 e^{s} + \operatorname{sen} rs)^2}} .$$

L7.

6. Usando a definção de integral dupla como limite de somas de Riemann, calcule a integral $\int_A f(x,y)dxdy$, sendo:

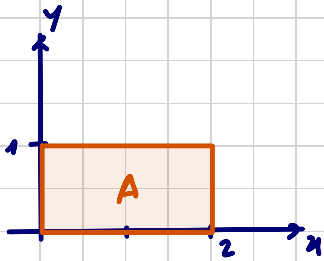
(a) $f(x,y) = x + 4y$, e A o bloco $[0,2] \times [0,1]$. (Resp.: 6)

(b) $f(x,y) = 3x^2 + 2y$, e A o bloco $[0,2] \times [0,1]$. (Resp.: 10)

(c) $f(x,y) = x^2 + 3y$, e A o bloco $[0,2] \times [1,5]$.

(b) $\int_A f = ?$ $\qquad f(x,y) = 3x^2 + 2y$

$A = [0,2] \times [0,1]$.

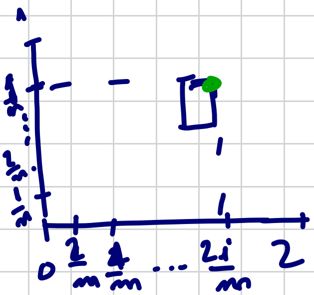


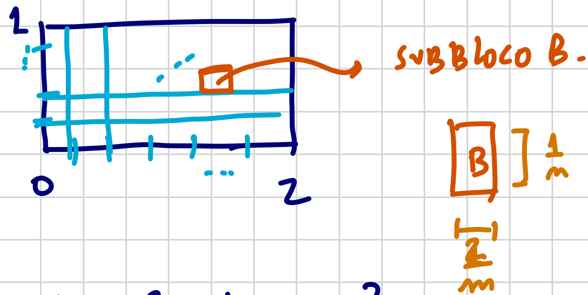
$$S(f, P) = \sum_{B \in P} M_i \cdot Vol(B) .$$

Seja $P = P_1 \times P_2$ uma partição regular de A, onde P_1 divide $[0, 2]$ em n subintervalos de comprimento $\Delta x = \frac{2-0}{n} = \frac{2}{n}$;

e P_2 divide $[0,1]$ em n subintervalos de comprimento

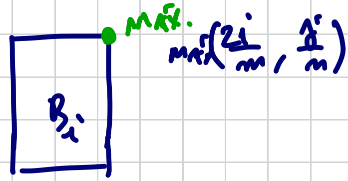
$$\Delta y = \frac{1-0}{n} = \frac{1}{n} .$$





$$Vol(B) = \frac{2}{n} \cdot \frac{1}{n} = \frac{2}{n \cdot n}$$

A função $f(x,y) = 3x^2 + 2y$ é crescente em A. Então o supremo (máximo) em cada subbloco i ocorre no canto superior direito:



Assim:

$$M_i = f(x,y)\Big|_{\substack{x=\frac{2i}{m}\\ y=\frac{j}{n}}} = 3\cdot\left(\frac{2i}{m}\right)^2 + 2\frac{j}{n} = \frac{12\,i^2}{m^2} + \frac{2j}{n}$$

Disso:

$$S(f;P) = \sum_{B\in P} M_i \cdot \operatorname{Vol}(B) =$$

$$= \sum_{j=1}^{n}\sum_{i=1}^{m}\left(\frac{12\,i^2}{m^2} + \frac{2j}{n}\right)\cdot\frac{2}{m\cdot n} =$$

$$= \sum_{j=1}^{n}\sum_{i=1}^{m}\left(\frac{24\,i^2}{m^3 n} + \frac{4j}{m\,n^2}\right) =$$

$$= \sum_{j=1}^{n}\sum_{i=1}^{m}\frac{24\,i^2}{m^3 n} + \sum_{j=1}^{n}\sum_{i=1}^{m}\frac{4j}{m\,n^2} =$$

$$= \frac{24}{m^3 n}\cdot\sum_{j=1}^{n}\underbrace{\sum_{i=1}^{m} i^2}_{\frac{m(2m+1)(m+1)}{6}} + \frac{4}{m\,n^2}\sum_{j=1}^{n}\sum_{i=1}^{m} j$$

$$= \frac{\overset{4}{\cancel{24}}}{\overset{2}{\cancel{m^3}}\,n}\;\frac{\cancel{m}(2m+1)(m+1)}{\cancel{6}}\underbrace{\sum_{j=1}^{n} 1}_{n} + \frac{4}{m\,n^2}\underbrace{\sum_{j=1}^{n} j}_{\frac{n(n+1)}{2}}\cdot\underbrace{\sum_{i=1}^{m} 1}_{m} =$$

$$= \frac{4}{m^2\cdot \cancel{n}}\cdot\frac{(2m+1)\cdot(m+1)}{1}\cdot\cancel{n} + \frac{\overset{2}{\cancel{4}}}{\cancel{m}\cdot n^{\cancel{2}}}\cdot\frac{\cancel{n}(n+1)}{\cancel{2}}\cdot\cancel{m}$$

$$= 4\cdot\frac{2m+1}{m}\cdot\frac{m+1}{m} + \frac{2(n+1)}{n}$$

$$\Rightarrow \overline{\int_A} f = \lim_{m,n\to\infty} 4\left(2+\underbrace{\frac{1}{m}}_{\to 0}\right)\cdot\left(1+\underbrace{\frac{1}{m}}_{\to 0}\right)+2+\underbrace{\frac{2}{n}}_{\to 0}$$

$$= 8 + 2 = 10$$

Analogamente se faz $\underline{\int_A} f = 10$

Portanto, $\int_A f = 10$.

Obs.: conferindo com integral iterada:

$$\int_{y=0}^{y=1}\int_{x=0}^{x=2}(3x^2+2y)\,dx\,dy =$$

$$= \int_{y=0}^{y=1}(x^3+2xy)\Big|_{x=0}^{x=2}dy = \int_0^1(8+4y)\,dy$$

$$(8y+2y^2)\Big|_0^1 = 8+2 = 10$$