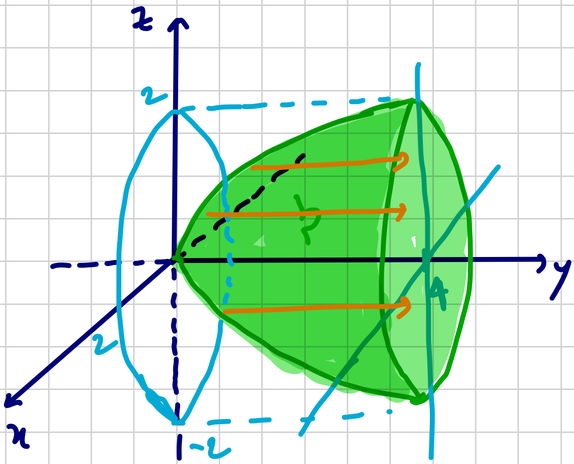
CÁLCULO 3, TURMA T2

26/06/26 – AULA 36

Na aula passada iniciamos o estudo de integrais triplas. Seguindo nos exemplos:

03) Calcule $\iiint_{\Omega} \sqrt{x^2+z^2}\, dV$, onde Ω é a região limitada pelo parabolóide $y = x^2+z^2$ e pelo plano $y=4$.

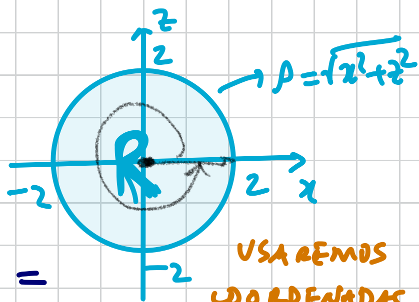
Solução: A região Ω é dada pela ilustração:



NOTE QUE, QUANDO TEMOS $Y=4$, ENTÃO

$x^2+z^2=4$

(circunf. de raio 2 no plano xz).



Assim, teremos:

$$\iiint_{\Omega} \sqrt{x^2+z^2}\, dV = \iint_{R} dx\,dz \int_{y=x^2+z^2}^{y=4} \underbrace{\sqrt{x^2+z^2}}_{\text{CONSTANTE PARA } y}\, dy =$$

USAREMOS COORDENADAS POLARES NO PLANO xz

$$= \iint_R \sqrt{x^2+z^2}\, dx\, dz \cdot \int_{y=x^2+z^2}^{y=4} dy = \iint_R \sqrt{x^2+z^2}\, dx\, dz \; y\Big|_{y=x^2+z^2}^{y=4} =$$

$$= \iint_R \sqrt{x^2+z^2}\, dx\, dz \cdot \left(4-(x^2+z^2)\right) =$$

$$= \iint_R \underbrace{\sqrt{x^2+z^2}}_{\rho} \cdot \left(4-\underbrace{(x^2+z^2)}_{\rho^2}\right) \underbrace{dx\, dz}_{\rho\, d\rho\, d\theta} =$$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{\rho=0}^{\rho=2} \rho \cdot (4-\rho^2) \cdot \rho\, d\rho\, d\theta = \int_{\theta=0}^{\theta=2\pi} d\theta \cdot \int_{\rho=0}^{\rho=2} (4\rho^2-\rho^4)\, d\rho =$$

$$= \theta\Big|_0^{2\pi} \cdot \left(\frac{4\rho^3}{3} - \frac{\rho^5}{5}\right)\Big|_0^2 = 2\pi \cdot \left(\frac{32}{3} - \frac{32}{5} - 0\right) =$$

$$= 2\pi \cdot \left(\frac{160-96}{15}\right) = 2\pi \cdot \frac{64}{15} = \frac{128\pi}{15} \text{ u.v.}$$

unidades de volume.

MUDANÇA GERAL DE VARIÁVEIS NO $\mathbb{R}^3$

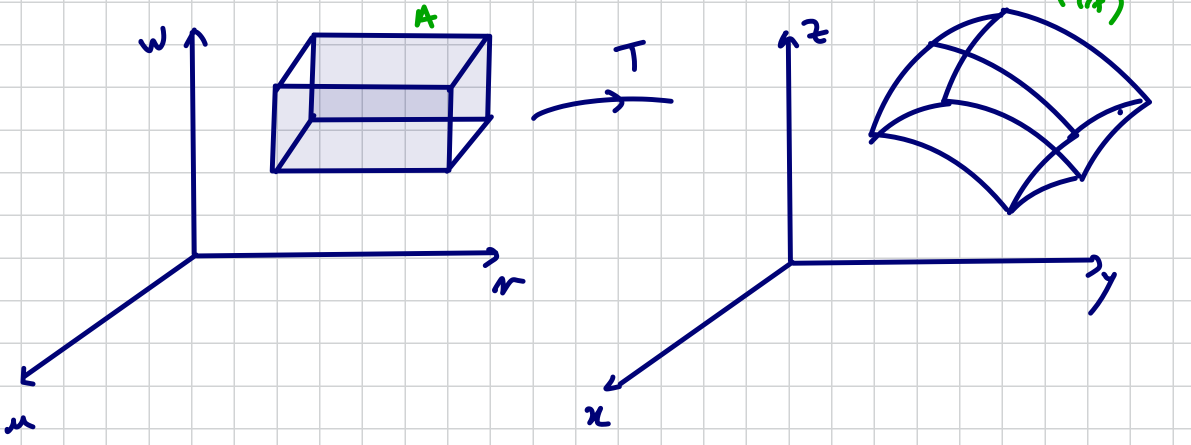
Da mesma forma que fizemos no $\mathbb{R}^2$, vamos deduzir uma fórmula para mudar de variáveis a fim de tornar possível o cálculo de uma integral tripla, aparentemente impossível pelas coordenadas dadas.

Seja $f: \Omega \subset \mathbb{R}^3 \to \mathbb{R}$ uma função integrável, e considere $T: \Omega \subset \mathbb{R}^3 \to \Omega' = T(\Omega) \subset \mathbb{R}^3$ uma transformação injetiva, de classe C^1 (ou seja, com com derivadas parciais contínuas), do sistema xyz para o sistema uvw, dado por:

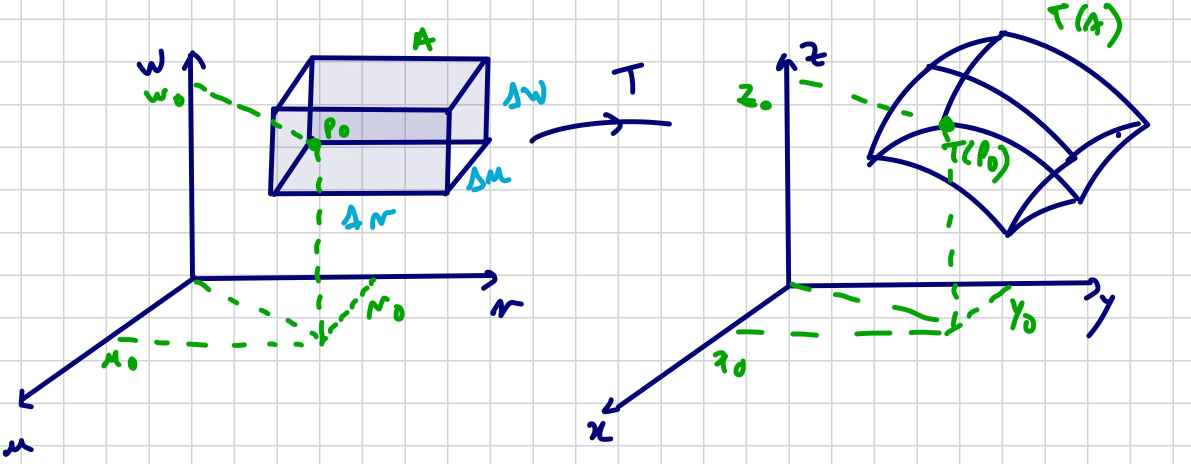
$$T(x,y,z) = (u,v,w),$$

onde:
$$\begin{cases} x = x(u,v,w) \\ y = y(u,v,w) \\ z = z(u,v,w) \end{cases}$$

Seja A um bloco no sistema uvw, conforme o esquema; onde T(A) no plano xyz fornece um "bloco retorcido".

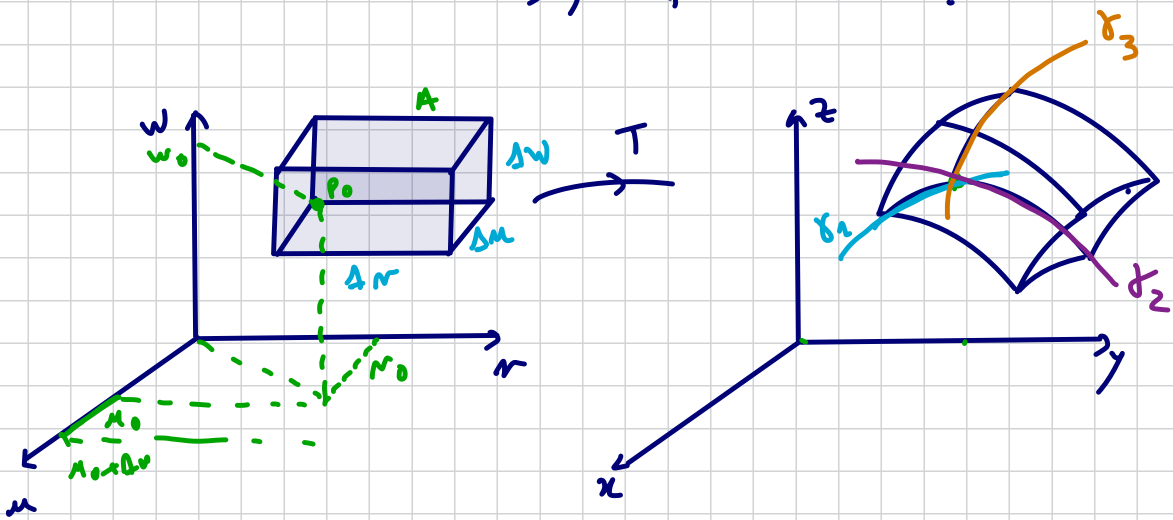


Seja $P_0(u_0, v_0, w_0)$ o vértice no canto inferior esquerdo, c.f. ilustração.



Sejam Δu, Δv e Δw as arestas do bloco A.

Considere γ_1, γ_2 e γ_3 as curvas no sistema xyz, passando pelo ponto $T(P_0)$, suportes às arestas do bloco $T(A)$, c.f. ilustração:



$$\gamma_1 : [u_0, u_0 + \Delta u] \longrightarrow \mathbb{R}^3$$

$$\gamma_1(u) = \big(x(u, v_0, w_0),\ y(u, v_0, w_0),\ z(u, v_0, w_0)\big)$$

$$\gamma_2 : [v_0, v_0 + \Delta v] \rightarrow \mathbb{R}^3$$

$$\gamma_2(v) = \big(x(u_0, v, w_0),\ y(u_0, v, w_0),\ z(u_0, v, w_0)\big)$$

$$\gamma_3 : [w_0, w_0 + \Delta w] \rightarrow \mathbb{R}^3$$

$$\gamma_3(w) = \big(x(u_0, v_0, w),\ y(u_0, v_0, w),\ z(u_0, v_0, w)\big)$$

Sejam $\vec{v}_1$, $\vec{v}_2$ e $\vec{v}_3$ os respectivos vetores tangentes a γ_1, γ_2 e γ_3, no ponto $T(P_0)$:

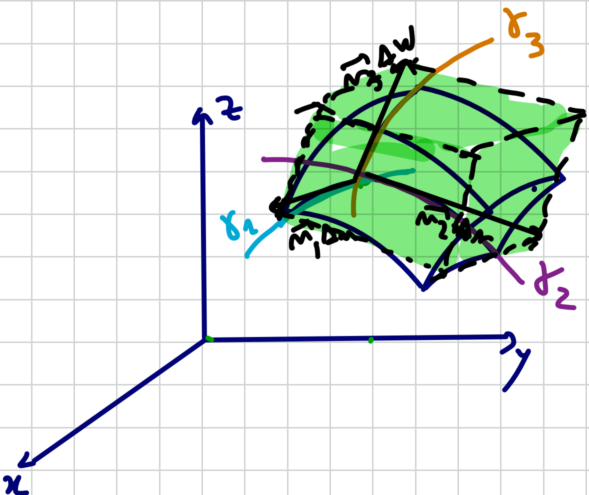
$$\vec{v}_1 = \gamma_1'(u_0) = \left(\frac{\partial x}{\partial u}(u_0,v_0,w_0), \frac{\partial y}{\partial u}(u_0,v_0,w_0), \frac{\partial z}{\partial u}(u_0,v_0,w_0)\right)$$

$$\vec{v}_2 = \gamma_2'(v_0) = \left(\frac{\partial x}{\partial v}(u_0,v_0,w_0), \frac{\partial y}{\partial v}(u_0,v_0,w_0), \frac{\partial z}{\partial v}(u_0,v_0,w_0)\right)$$

$$\vec{v}_3 = \gamma_3'(w_0) = \left(\frac{\partial x}{\partial w}(u_0,v_0,w_0), \frac{\partial y}{\partial w}(u_0,v_0,w_0), \frac{\partial z}{\partial w}(u_0,v_0,w_0)\right)$$

Considere o paralelepípedo formado pelos vetores $\vec{v}_1 \Delta u$, $\vec{v}_2 \cdot \Delta v$ e $\vec{v}_3 \cdot \Delta w$:

$$\hookrightarrow \left(\frac{\partial x}{\partial w}\Delta w, \frac{\partial y}{\partial w}\Delta w, \frac{\partial z}{\partial w}\Delta w\right)$$



Da ALGA, o volume V de um paralelepípedo formado por três vetores é dado pelo módulo do produto misto entre eles.

Ou seja;

$$V = \left| [\vec{n}_1 \Delta u, \vec{n}_2 \Delta v, \vec{n}_3 \Delta w] \right| \text{; onde:}$$

$$[\vec{n}_1 \Delta u, \vec{n}_2 \Delta v, \vec{n}_3 \Delta w] = \begin{vmatrix} \frac{\partial x}{\partial u}\Delta u & \frac{\partial y}{\partial u}\Delta u & \frac{\partial z}{\partial u}\Delta u \\ \frac{\partial x}{\partial v}\Delta v & \frac{\partial y}{\partial v}\Delta v & \frac{\partial z}{\partial v}\Delta v \\ \frac{\partial x}{\partial w}\Delta w & \frac{\partial y}{\partial w}\Delta w & \frac{\partial z}{\partial w}\Delta w \end{vmatrix}$$

$$= \underbrace{\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix}}_{\det J(T)(u,v,w)} \cdot \Delta u \, \Delta v \, \Delta w$$

$$= \det J(T)(u,v,w) \cdot \Delta u \cdot \Delta v \cdot \Delta w$$

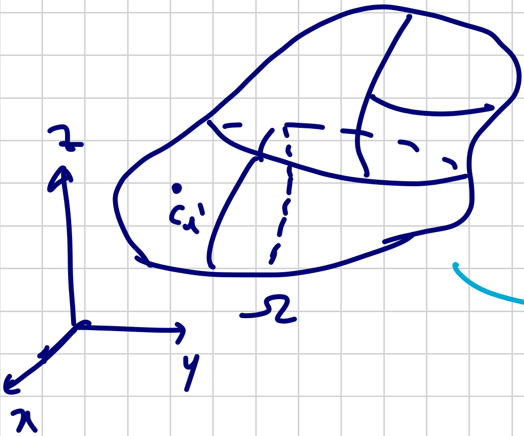
Assim:

$$V = \left| \det J(T)(u,v,w) \right| \cdot \underbrace{\Delta u}_{>0} \cdot \underbrace{\Delta v}_{>0} \cdot \underbrace{\Delta w}_{>0}$$

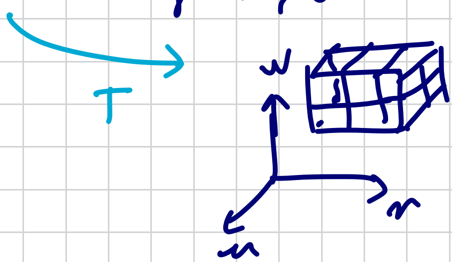
Assim, o volume do bloco retorcido no sistema xyz é aproximadamente o volume do paralelepípedo, dado por:

$$V = |\det J(T)(u,v,w)| \cdot \Delta u \cdot \Delta v \cdot \Delta w.$$

Seja $f: \Omega \subset \mathbb{R}^3 \to \mathbb{R}$ integrável.
Seja P uma partição de Ω.



Pela $T: \Omega \to \Omega'$ vamos obter uma partição em blocos:



Sendo c_i um ponto qualquer em cada subbloco, montando a soma de Riemann:

$$V \approx \sum_{i=1}^{n} f(c_i) \cdot Vol(B_i).$$

$$= \sum_{i=1}^{n} f(c_i) \cdot \left| \det J(T)(u_i, v_i, w_i) \right| \cdot \Delta u_i \, \Delta v_i \, \Delta w_i$$

Tomando o limite com $n \to \infty$, vamos obter:

$$\boxed{\begin{aligned} &\iiint_{\Omega} f(x,y,z)\,dx\,dy\,dz = \\ &= \iiint_{\Omega'} f(x(u,v,w), y(u,v,w), z(u,v,w)) \cdot \left| \det J(T)(u,v,w) \right| \cdot du\,dv\,dw \end{aligned}}$$

FÓRMULA DE MUDANÇA DE VARIÁVEIS NO $\mathbb{R}^3$.

Vejamos um EXEMPLO:

Ex.: Calcule $\displaystyle\iiint_{\Omega} \frac{e^{x-y+z}}{x+y-z}\,dx\,dy\,dz$, onde Ω é a região do $\mathbb{R}^3$ dada por:

$$0 \leq x - y + z \leq 1$$

$$1 \leq x + y - z \leq 2$$

$$0 \leq z \leq 1.$$

Soluție: Scriem $\begin{cases} u = x - y + z \\ v = x + y - z \\ w = z \end{cases} \longrightarrow z = w$

$$
\begin{aligned}
& u = x \cancel{- y} \cancel{+ z} \\
+ \; & v = x \cancel{+ y} \cancel{- z} \\
\hline
& 2x = u + v \Rightarrow \boxed{x = \tfrac{1}{2} u + \tfrac{1}{2} v}
\end{aligned}
$$

$$v = x + y - z \Rightarrow y = v - x + z$$

$$y = v - \left(\tfrac{1}{2} u + \tfrac{1}{2} v\right) + w$$

$$\boxed{y = -\tfrac{1}{2} u + \tfrac{1}{2} v + w}$$

$$J(T)(u,v,w) = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \det J(T)(u,v,w) = \begin{vmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 1 \end{vmatrix} \begin{matrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{matrix}$$

$$- \quad - \quad - \quad + \quad + \quad +$$

$$= \frac{1}{4} + 0 + 0 - 0 - 0 + \frac{1}{4} = \frac{1}{2} //$$

Além disso; pelas condições:

$$\left.\begin{array}{l} 0 \leq u \leq 1 \\ 1 \leq v \leq 2 \\ 0 \leq w \leq 1 \end{array}\right\} \Omega'$$

Assim, obtemos:

$$\iiint_{\Omega} \frac{e^{x-y+z}}{x+y-z}\, dx\, dy\, dz = \iiint_{\Omega'} \frac{e^{u}}{v} \left|\frac{1}{2}\right| . du\, dv\, dw$$

$$= \frac{1}{2} . \int_{w=0}^{w=1} \int_{v=0}^{v=2} \int_{u=0}^{u=1} \frac{e^{u}}{v}\, du\, dv\, dw =$$

$$= \frac{1}{2} . \int_{w=0}^{w=1} dw . \int_{v=1}^{v=2} \frac{dv}{v} \int_{u=0}^{u=1} e^{u} du =$$

$$= \frac{1}{2}\, w \Big|_{0}^{1} . \ln|v| \Big|_{1}^{2} . e^{u} \Big|_{0}^{1} =$$

$$= \frac{1}{2} \cdot (1-0) \cdot (\ln 2 - \underbrace{\ln 1}_{=0}) \cdot (e^1 - e^0) =$$

$$= \frac{1}{2} \cdot 1 \cdot \ln 2 \cdot (e-1) = \frac{e-1}{2} \cdot \ln 2$$