CÁLCULO 3, TURMA T2.

08/06/26 - AULA 29

LISTA 08

2. Calcule cada integral dupla a seguir:

(a) $\int_0^3 \int_0^2 (4-y^2)dydx$

(b) $\int_0^3 \int_{-2}^0 (x^2y - 2xy)dydx$

(c) $\int_\pi^{2\pi} \int_0^\pi (\operatorname{sen} x + \cos y)dxdy$

(d) $\int_1^2 \int_y^{y^2} dxdy$

(e) $\int_0^1 \int_0^{y^2} 3y^3 e^{xy} dxdy$

(f) $\int_1^4 \int_0^{\sqrt{x}} \frac{3}{2} e^{\frac{y}{\sqrt{x}}} dydx$

$$(e) \int_{y=0}^{y=1} \int_{x=0}^{x=y^2} 3y^3 e^{xy} dx\, dy = \int_{y=0}^{y=1} 3\cancel{y^3}^{\,2} \cdot \left(\int_{x=0}^{x=y^2} e^{\overbrace{xy}^{r}} \underbrace{y\,dx}_{dr} \right) dy =$$

$$\int e^r dr = e^r + C$$

$$r = xy$$

$$\Rightarrow dr = y \cdot e^{xy}$$

$$= \int_{y=0}^{y=1} 3y^2 \cdot e^{xy} \Big|_{x=0}^{x=y^2} dy = \int_{y=0}^{y=1} 3y^2 \left(e^{y^3} - \underbrace{e^0}_{=1} \right) dy =$$

$$= \int_0^1 3y^2 \cdot \underbrace{e^{y^3}}_{e^r} dy - \int_0^1 3y^2 \cdot dy =$$

$$u = y^3$$
$$du = 3y^2\,dy$$

$$= e^{y^3}\Big|_0^1 - y^3\Big|_0^1 = e^1 - e^0 - [1^3 - 0^3]$$

$$= e - 1 - 1 = e - 2$$

$$\text{(f)} \int_{x=1}^{x=4}\int_{y=0}^{y=\sqrt{x}} \frac{3}{2}\cdot e^{\frac{y}{\sqrt{x}}}\cdot dy\,dx = \frac{3}{2}\int_{x=1}^{x=4} \sqrt{x}\int_{y=0}^{y=\sqrt{x}} e^{\frac{y}{\sqrt{x}}}\frac{1}{\sqrt{x}}\,dy\,dx$$

$$\int e^u\,du = e^u + c$$

$$u = \frac{y}{\sqrt{x}} \Rightarrow du = \frac{1}{\sqrt{x}}\,dy$$

$$= \frac{3}{2}\int_{x=1}^{x=4} \sqrt{x}\cdot e^{\frac{y}{\sqrt{x}}}\Big|_{y=0}^{y=\sqrt{x}}\,dx = \frac{3}{2}\int_{x=1}^{x=4} \sqrt{x}\cdot\left(e^{\frac{\sqrt{x}}{\sqrt{x}}} - e^{\frac{0}{\sqrt{x}}}\right)dx$$

$$= \frac{3}{2}\int_1^4 \sqrt{x}\cdot(e-1)\,dx = \frac{3(e-1)}{2}\int_1^4 x^{\frac{1}{2}}\,dx =$$

$$= \frac{3(e-1)}{2} \cdot \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\Bigg|_1^4 = \cancel{\frac{3}{2}}(e-1) \cdot \frac{\cancel{2}}{\cancel{3}}\, x^{\frac{3}{2}}\Bigg|_1^4 =$$

$$= (e-1) \cdot \sqrt{x^3}\,\Bigg|_1^4 = (e-1)\, x\sqrt{x}\,\Bigg|_1^4 =$$

$$(e-1) \cdot \left[4\sqrt{4} - 1\sqrt{1}\right] = (e-1) \cdot (8-1) =$$

$$= 7(e-1)$$

L8

1. Calcule cada integral dupla a seguir:

(a) $\displaystyle\iint_A \operatorname{sen}(x+y)\,dxdy$, onde $A = [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$.

(b) $\displaystyle\iint_A \frac{xy^2}{1+x^2}\,dxdy$, onde $A = \{(x,y) \in \mathbb{R}^2 \,:\, 0 \le x \le 2 \text{ e } -3 \le y \le 3\}$.

(c) $\displaystyle\iint_A x \operatorname{sen} xy\,dxdy$, onde A é o retângulo $0 \le x \le \pi$, $0 \le y \le 1$. (Resp. π)

(c) $$\int_{y=0}^{y=1}\int_{x=0}^{x=\pi} x \cdot \operatorname{sen} xy\, dx\, dy = \int_{x=0}^{x=\pi}\int_{y=0}^{y=1} \underbrace{x \operatorname{sen} xy\, dy}_{dv}\, dx =$$

$\int \operatorname{sen} v\, dv = -\cos v + c$

$v = xy$

$\Rightarrow dv = x\, dy$

$$= \int_{x=0}^{x=\pi} \left(\int_{y=0}^{y=1} \operatorname{sen} \underbrace{xy}_{u} \cdot \underbrace{(x\,dy)}_{du} \right) dx =$$

$$= \int_{x=0}^{x=\pi} (-\cos xy) \Big|_{y=0}^{y=1} dx = \int_{x=0}^{x=\pi} (-\cos x + \underbrace{\cos 0}_{=1}) dx =$$

$$\int_0^{\pi} -\cos x \, dx + \int_0^{\pi} dx = -\operatorname{sen} x \Big|_0^{\pi} + x \Big|_0^{\pi} =$$

$$= -[\underbrace{\operatorname{sen} \pi}_{=0} - \underbrace{\operatorname{sen} 0}_{=0}] + [\pi - 0] = 0 + \pi = \pi$$

LISTA 05:

14. Seja $f(x, y) = x^2 - y^2$. Represente geometricamente $\nabla f(x_o, y_o)$, sendo

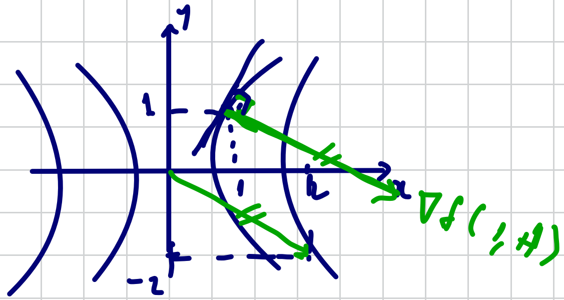
(a) $(x_o, y_o) = (1, 1)$. (b) $(x_o, y_o) = (-1, 1)$. (c) $(x_o, y_o) = (-1, -1)$.

(a) $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = (2x, -2y)$

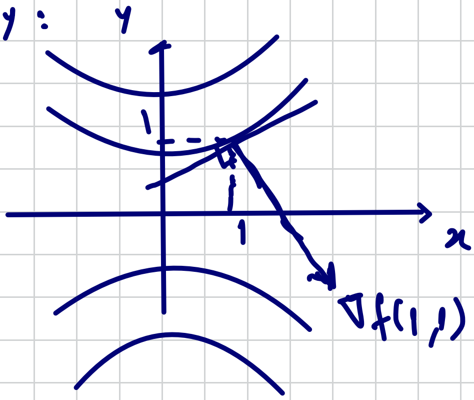
$\nabla f(1,1) = (2 \cdot 1, -2 \cdot 1) = (2, -2)$

Olhando curvas de nível: $z = x^2 - y^2$

- $z = k$, $k > 0$ - hipérboles com eixo real sendo o eixo x:



- $z = k$; $k < 0$: hipérboles com eixo real sendo o y:



L5:

12. Em cada item, obtenha a derivada direcional no ponto P, segundo a direção θ indicada.

(a) $f(x, y) = \ln\sqrt{x^2 + y^2}$, $P(2, 1)$, $\theta = 60^o$.

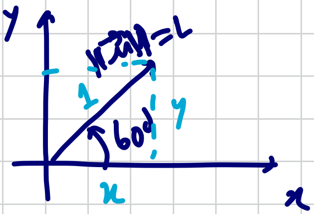
(b) $f(x, y) = e^x \cos y$, $P(0, 0)$, $\theta = 60^o$.

(c) $f(x, y) = \ln(x^2 + y^2)$, $P(1, 1)$, $\theta = 45^o$.

(a) $f(x,y) = \ln\sqrt{x^2+y^2} = \ln(x^2+y^2)^{\frac{1}{2}} = \frac{1}{2}\ln(x^2+y^2)$

$\frac{\partial f}{\partial \vec{u}}(P) = ?$ onde $\vec{u}$ tem direção $\theta = 60^o$.

$\|\vec{u}\| = 1$.



$\cos 60^o = \frac{x}{1} \Rightarrow x = \cos 60^o$

$\Rightarrow x = \frac{1}{2}$

$\text{sen}\, 60^o = \frac{y}{1} \Rightarrow y = \text{sen}\, 60^o$

$y = \frac{\sqrt{3}}{2}$

$\Rightarrow \boxed{\vec{u} = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)}$

note que $\|\vec{u}\| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$

$\|\vec{u}\| = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$.

Assim, teremos:

$\frac{\partial f}{\partial \vec{u}} = \nabla f \cdot \vec{u}$; onde:

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \left(\frac{1}{\cancel{2}} \cdot \frac{\cancel{2}x}{x^2+y^2}, \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}y}{x^2+y^2}\right)$$

$$(\ln r)' = \frac{r'}{r} \qquad \Rightarrow \nabla f = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$$

Avem:

$$\frac{\partial f}{\partial \vec{n}} = \nabla f \cdot \vec{n} = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right) \cdot \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

$$= \frac{x}{x^2+y^2} \cdot \frac{1}{2} + \frac{y}{x^2+y^2} \cdot \frac{\sqrt{3}}{2} =$$

$$= \frac{x+\sqrt{3}\,y}{2(x^2+y^2)}.$$

In fine:

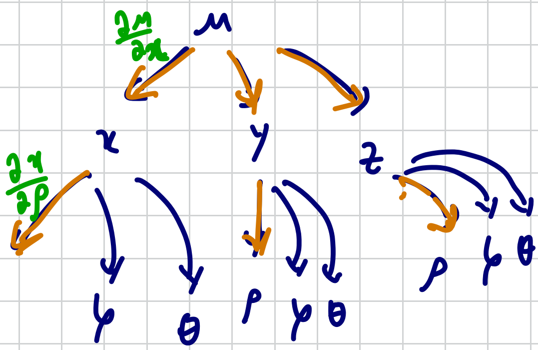
$$\frac{\partial f}{\partial \vec{n}}(2,1) = \frac{2+\sqrt{3}\cdot 1}{2(2^2+1^2)} = \frac{2+\sqrt{3}}{10} \quad //$$

L5

8. Suponha que $u = f(x,y,z)$ seja diferenciável, onde $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$ e $z = \rho\cos\phi$. Calcule $\frac{\partial u}{\partial \rho}$, $\frac{\partial u}{\partial \phi}$ e $\frac{\partial u}{\partial \theta}$ em termos das derivadas parciais em x, y e z.

$$u = f(x, y, z)$$

$$\begin{cases} x = \rho \operatorname{sen}\varphi \cos\theta \\ y = \rho \operatorname{sen}\varphi \cdot \operatorname{sen}\theta \\ z = \rho \cos\varphi \end{cases}$$



$$\frac{\partial u}{\partial \rho} = ?$$

$$\frac{\partial u}{\partial \rho} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \rho} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \rho} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial \rho}\ ;\ \text{onde:}$$

$$\frac{\partial x}{\partial \rho} = \operatorname{sen}\varphi\cdot\cos\theta\ ;\quad \frac{\partial y}{\partial \rho} = \operatorname{sen}\varphi\cdot\operatorname{sen}\theta\ ;\quad \frac{\partial z}{\partial \rho} = \cos\varphi$$

Disso, obtemos:

$$\boxed{\frac{\partial u}{\partial \rho} = \frac{\partial u}{\partial x}\cdot\operatorname{sen}\varphi\cdot\cos\theta + \frac{\partial u}{\partial y}\cdot\operatorname{sen}\varphi\cdot\operatorname{sen}\theta + \frac{\partial u}{\partial z}\cdot\cos\varphi}$$

O mesmo faz-se para as outras duas.

$$\frac{\partial u}{\partial \varphi} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \varphi} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \varphi} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial \varphi} \quad ; \text{ onde:}$$

$$x = \rho \operatorname{sen}\varphi . \cos\theta \Rightarrow \frac{\partial x}{\partial \varphi} = \rho \cos\varphi . \cos\theta .$$

$$y = \rho \operatorname{sen}\varphi . \operatorname{sen}\theta \Rightarrow \frac{\partial y}{\partial \varphi} = \rho \cos\varphi . \operatorname{sen}\theta .$$

$$z = \rho \cos\varphi \Rightarrow \frac{\partial z}{\partial \varphi} = -\rho \operatorname{sen}\varphi .$$

Então:

$$\boxed{\frac{\partial u}{\partial \varphi} = \frac{\partial u}{\partial x} . \rho \cos\varphi \cos\theta + \frac{\partial u}{\partial y} . \rho \cos\varphi . \operatorname{sen}\theta - \frac{\partial u}{\partial z} . \rho \operatorname{sen}\varphi}$$

Analogamente faz-se para obter $\frac{\partial u}{\partial \theta}$.