CÁLCULO 3, TURMAS T_2 e T_3 | 22/06/26 – AULA 34

INTEGRAIS IMPRÓPRIAS

Analogamente ao estudado no cálculo 2, estudamos integrais onde a região a ser integrada é ilimitada (o que corresponde na reta a um dos limites de integração ser infinito); ou dentro da região de integração existir uma singularidade (ou seja, um ponto no qual a função está definida)

Assim, separamos as integrais impróprias em dois tipos:

1.º TIPO: Quando existe uma singularidade em Ω. (Ω limitado).

Por exemplo, se f não estiver definida em um ponto x_0 em Ω, e tivermos:

$$\iint_{\Omega} f \quad ;$$

define-se $\tilde{\Omega} = \Omega \setminus B_{\varepsilon}(x_0)$, $\varepsilon > 0$.

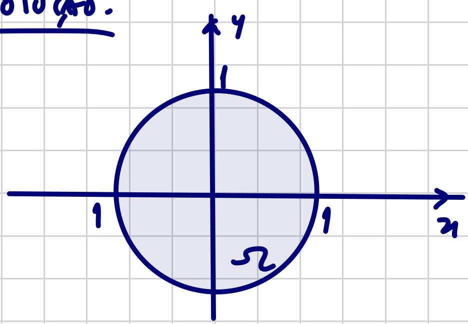
Disso:

$$\iint_{\Omega} f = \lim_{\varepsilon \to 0} \iint_{\tilde{\Omega}} f = \lim_{\varepsilon \to 0} \iint_{\Omega \setminus B_\varepsilon(x_0)} f .$$

$\tilde{\Omega} = \Omega \setminus B_\varepsilon(x_0)$.

Ex.: Calcule $\displaystyle\iint_{x^2+y^2 \le 1} \frac{dx\,dy}{\sqrt{x^2+y^2}}$.

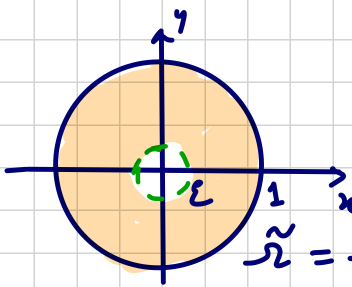
Solução:



NOTE QUE f NÃO ESTÁ DEFINIDA NA ORIGEM $(0,0) \in \Omega$.

ENTÃO, CONSIDERE $\varepsilon > 0$.
$(\varepsilon < 1)$

E SEJA $B_\varepsilon(0)$



Vamos usar coordenadas polares para resolver.

$\tilde{\Omega} = \Omega \setminus B_\varepsilon(0)$

$$\iint\limits_{x^2+y^2\le 1} \frac{dx\,dy}{\sqrt{x^2+y^2}} = \lim_{\varepsilon\to 0^+} \iint\limits_{\tilde{\Omega}} \frac{dx\,dy}{\sqrt{x^2+y^2}} =$$

$$\left[\begin{array}{l} dx\,dy = \rho\,d\rho\,d\theta \\ \rho = \sqrt{x^2+y^2} \end{array}\right.$$

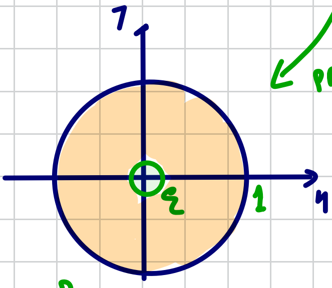
$$= \lim_{\varepsilon\to 0^+} \int_{\theta=0}^{\theta=2\pi} \int_{\rho=\varepsilon}^{\rho=1} \frac{\cancel{\rho}\,d\rho\,d\theta}{\cancel{\rho}} =$$

$$= \lim_{\varepsilon\to 0^+} \int_{\theta=0}^{\theta=2\pi} d\theta \cdot \int_{\rho=\varepsilon}^{\rho=1} d\rho = \lim_{\varepsilon\to 0^+} \theta\Big|_0^{2\pi} \cdot \rho\Big|_\varepsilon^1 =$$

$$= \lim_{\varepsilon\to 0^+} 2\pi\cdot(1-\varepsilon) = 2\pi(1-0) = 2\pi$$

Obs: Quando existir o limite, dizemos que a integral imprópria converge para o valor encontrado. Quando não existir o limite ou quando for infinito, dizemos que a integral imprópria diverge.

02) $$\iint_{x^2+y^2\le 1} \ln\left(\frac{1}{x^2+y^2}\right)dx\,dy = \lim_{\varepsilon\to 0^+}\int_{\theta=0}^{\theta=2\pi}\int_{\rho=\varepsilon}^{\rho=1}\ln\left(\frac{1}{\rho^2}\right)\cdot\rho\,d\rho\,d\theta =$$



Ω_ε

PRECISAMOS RETIRAR A ORIGEM.

(usando coord. polares)

$dx\,dy = \rho\,d\rho\,d\theta.$

$x^2+y^2=\rho^2$

$$= \lim_{\varepsilon\to 0^+}\int_{\theta=0}^{\theta=2\pi}d\theta\cdot\int_{\rho=\varepsilon}^{\rho=1}\ln(\rho^{-2})\,\rho\,d\rho$$

$$= \lim_{\varepsilon\to 0^+} 2\pi\cdot\int_{\rho=\varepsilon}^{\rho=1}-2\cdot\ln\rho\cdot\rho\,d\rho\cdot =$$

$\ln a^m = m\cdot\ln a$

$$= -4\pi\cdot\lim_{\varepsilon\to 0^+}\underbrace{\int_{\rho=\varepsilon}^{\rho=1}\ln\rho\cdot\rho\,d\rho}_{\text{integrar por partes.}} =$$

$$\int\rho\cdot\ln\rho\,d\rho = \int u\,dv = u\cdot v - \int v\,du$$

$$u=\ln\rho \Rightarrow du = \frac{1}{\rho}\cdot d\rho$$

$$dv = \rho\, d\rho \Rightarrow v = \int \rho\, d\rho = \frac{\rho^2}{2}$$

$$\Rightarrow \int \rho \cdot \ln\rho \cdot d\rho = \ln\rho \cdot \frac{\rho^2}{2} - \int \frac{\rho^{\cancel{2}}}{2} \cdot \frac{1}{\cancel{\rho}}\, d\rho$$

$$= \frac{\rho^2}{2} \cdot \ln\rho - \frac{1}{2}\int \rho\, d\rho = \frac{\rho^2}{2}\ln\rho - \frac{\rho^2}{4}$$

$$= -4\pi \cdot \lim_{\varepsilon \to 0^+} \left(\frac{\rho^2}{2} \cdot \ln\rho - \frac{\rho^2}{4} \right)\Bigg|_{\varepsilon}^{1} =$$

$$= -4\pi \cdot \lim_{\varepsilon \to 0^+} \frac{1}{2} \cdot \underbrace{\ln 1}_{0} - \frac{1}{4} - \frac{\varepsilon^2}{2}\ln\varepsilon + \frac{\varepsilon^2}{4}$$

$$= -4\pi \cdot \lim_{\varepsilon \to 0^+} \left(-\frac{1}{4} \cdot - \frac{\varepsilon^2}{2}\ln\varepsilon + \frac{\varepsilon^2}{4} \right)$$

$$= -4\pi \cdot \lim_{\varepsilon \to 0^+} \left(-\frac{1}{4} - \frac{1}{2} \underbrace{\frac{\ln\varepsilon}{\frac{1}{\varepsilon^2}}}_{\text{L'HOPITAL}} + \frac{\varepsilon^2}{4} \right)$$

$$= -4\pi \cdot \left(-\frac{1}{4} - \frac{1}{2} \cdot \lim_{\varepsilon \to 0^+} \frac{\ln\varepsilon}{\varepsilon^{-2}} + \lim_{\varepsilon \to 0^+} \frac{\varepsilon^2}{4} \right)$$

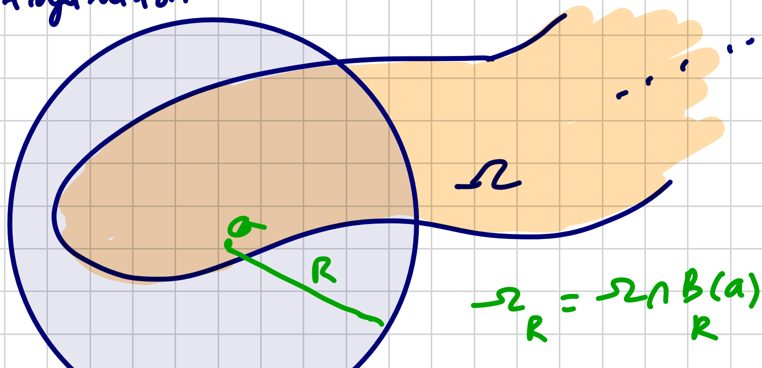
$$= -4\pi \cdot \left(-\frac{1}{4} - \frac{1}{2} \cdot \lim_{\varepsilon \to 0^+} \frac{\frac{1}{\varepsilon}}{-2\varepsilon^{-3}} + 0 \right)$$

$$= -4\pi \left(-\frac{1}{4} - \frac{1}{2} \lim_{\varepsilon \to 0^+} \frac{\frac{1}{\varepsilon}}{-\frac{2}{\varepsilon^2}} \right) =$$

$$= -4\pi \left(-\frac{1}{4} - \frac{1}{2} \cdot \underbrace{\lim_{\varepsilon \to 0^+} -\left(\frac{1}{\cancel{\varepsilon}} \cdot \frac{\varepsilon^2}{2} \right)}_{\to 0} \right) = -4\pi \left(-\frac{1}{4} \right) = \pi$$

2º TIPO: Quando a região Ω é ilimitada:

Neste caso, tomamos uma BOLA ou RETÂNGULO de medida (raio ou lado) que passaremos o limite indo ao infinito. Então, a região Ω ficará no seu interior com o raio (lado) tendendo ao infinito.



$\Omega_R = \Omega \cap B_R(a)$.

Assim $\Omega = \lim_{R\to+\infty} \Omega_R$

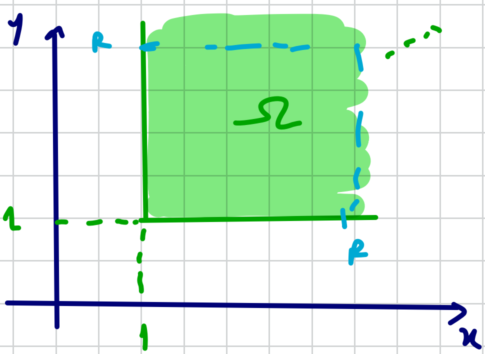
Então: $\iint_{\Omega} f = \lim_{R\to+\infty} \iint_{\Omega_R} f.$

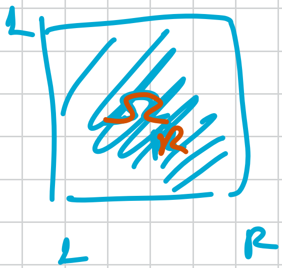
Vejamos exemplos:

01) $\int_1^{+\infty}\int_1^{+\infty} \frac{dx\,dy}{x^2y^2} = ?$

$f(x,y) = \frac{1}{x^2y^2}$ não está definida em $(0,0)$, não está em Ω:

$\Omega = [1,+\infty) \times [1,+\infty).$





Assim:

$$\int_1^{+\infty}\int_1^{+\infty} \frac{dx\,dy}{x^2y^2} = \lim_{R\to+\infty} \int_{y=1}^{y=R}\int_{x=1}^{x=R} \frac{dx\,dy}{x^2y^2} =$$

$$= \lim_{R\to+\infty} \int_{y=1}^{y=R} \frac{1}{y^2}\int_{x=1}^{x=R} \frac{1}{x^2}\,dx\,dy =$$

$$= \lim_{R\to+\infty} \left(\int_{y=1}^{y=R} y^{-2}\,dy \cdot \int_{x=1}^{x=R} x^{-2}\,dx \right) =$$

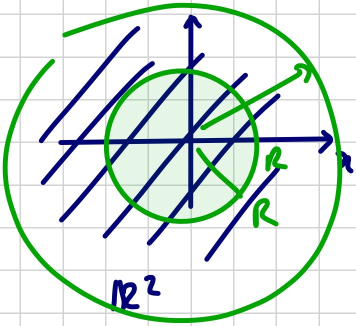
$$= \lim_{R\to+\infty} \left.\frac{y^{-1}}{-1}\right|_1^R \cdot \left.\frac{x^{-1}}{-1}\right|_1^R =$$

$$= \lim_{R\to+\infty} \left.\left(-\frac{1}{y}\right)\right|_1^R \cdot \left.\left(-\frac{1}{x}\right)\right|_1^R =$$

$$= \lim_{R\to+\infty} \left(-\frac{1}{R}+1\right)\cdot\left(-\frac{1}{R}+1\right) =$$

$$= \lim_{R\to+\infty} \left(1-\underbrace{\frac{1}{R}}_{\to 0}\right)^2 = 1$$

02) $$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-x^2-y^2}\,dx\,dy = ?$$



$\Omega = \mathbb{R}^2 =$

VAMOS TOMAR $\Omega_R = B_R(0)$

E FAZER $R \to +\infty$

Usaremos, novamente, coordenadas polares.

$$\int_{y=-\infty}^{y=+\infty}\int_{x=-\infty}^{x=+\infty} e^{-x^2-y^2}\,dx\,dy = \lim_{R\to+\infty} \iint_{B_R(0)} e^{-x^2-y^2}\,dx\,dy =$$

$$= \lim_{R\to+\infty} \int_{\theta=0}^{\theta=2\pi}\int_{\rho=0}^{\rho=R} e^{-\rho^2}\cdot\rho\,d\rho\,d\theta = \lim_{R\to+\infty} \int_{\theta=0}^{\theta=2\pi} d\theta \cdot \left(-\frac{1}{2}\right)\int_{\rho=0}^{\rho=R} e^{-\rho^2}\cdot(-2\rho\,d\rho)$$

$-\frac{1}{2}\int e^{-\rho^2}(-2\rho\,d\rho) =$

$u = -\rho^2 \Rightarrow du = -2\rho\,d\rho$

$$= -\frac{1}{2}\lim_{R\to+\infty} \theta\Big|_0^{2\pi} \cdot e^{-\rho^2}\Big|_0^R =$$

$$= -\frac{1}{2} \cdot \lim_{R \to +\infty} (2\pi - 0) \cdot \left(e^{-R^2} - e^0 \right)$$

$$= -\pi \cdot \lim_{R \to +\infty} \left(\underbrace{\frac{1}{e^{R^2}}}_{\to 0} - 1 \right) = -\pi \cdot (-1) = \pi$$

<u>conclusão:</u> $\displaystyle\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-x^2-y^2} \, dx \, dy = \pi$

<u>APLICAÇÃO DESSE EXEMPLO:</u> INTEGRAL DE POISSON:

<u>AF!</u> $\displaystyle\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}.$

De fato:

$$\pi = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-x^2-y^2} \, dx \, dy = \int_{-\infty}^{+\infty} e^{-x^2} dx \cdot \int_{-\infty}^{+\infty} e^{-y^2} dy =$$

DO EXEMPLO ACIMA

$$= \left(\int_{-\infty}^{+\infty} e^{-x^2} dx \right)^2$$

$$\Rightarrow \left(\int_{-\infty}^{+\infty} e^{-x^2}\,dx\right)^2 = \pi \quad \Rightarrow \quad \boxed{\int_{-\infty}^{+\infty} e^{-x^2}\,dx = \sqrt{\pi}}$$

LISTA 09:

13. Calcule as integrais impróprias abaixo.

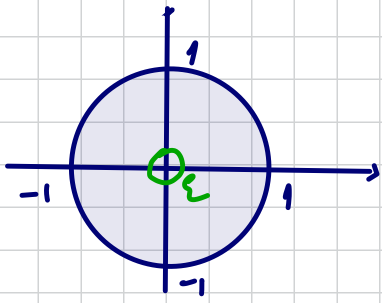
(a) $\displaystyle\int_0^\infty \int_0^\infty e^{-x-y}\,dx\,dy$ (b) $\displaystyle\int_0^\infty \int_0^\infty x^2 e^{-x^2-y^2}\,dx\,dy$

(c) $\displaystyle\iint_{x^2+y^2\le 1} \frac{x^2\,dx\,dy}{(x^2+y^2)^{\frac{7}{4}}}$ (d) $\displaystyle\iint_{x^2+y^2\le 1} \ln\sqrt{x^2+y^2}\,dx\,dy$

(c) $\displaystyle\iint_{x^2+y^2\le 1} \frac{x^2\,dx\,dy}{(x^2+y^2)^{\frac{7}{4}}}$

$\varepsilon > 0; \quad \varepsilon < 1$

$\Omega_\varepsilon = B_1(0) \setminus B_\varepsilon(0)$



$x^2+y^2 \neq 0$

$(x,y) \neq (0,0)$

$$\iint_{x^2+y^2\le 1} \frac{x^2\,dx\,dy}{(x^2+y^2)^{\frac{7}{4}}} = \lim_{\varepsilon\to 0^+} \int_{\theta=0}^{\theta=2\pi}\int_{\rho=\varepsilon}^{\rho=1} \frac{(\rho\cos\theta)^2\,\rho\,d\rho\,d\theta}{(\rho^{\cancel{2}})^{\frac{7}{\cancel{4}\,2}}} =$$

$$= \lim_{\varepsilon \to 0^+} \int_{\theta=0}^{\theta=2\pi} \int_{\rho=\varepsilon}^{\rho=1} \frac{\rho^2 \cos^2\theta \cdot \rho \, . \, d\rho \, d\theta}{\rho^{\frac{7}{2}}} =$$

$$= \lim_{\varepsilon \to 0^+} \int_{\theta=0}^{\theta=2\pi} \cos^2\theta \, d\theta \cdot \int_{\rho=\varepsilon}^{\rho=1} \rho^{3-\frac{7}{2}} \, d\rho$$

$$= \int_0^{2\pi} \cos^2\theta \, d\theta \cdot \lim_{\varepsilon \to 0^+} \int_\varepsilon^1 \rho^{-\frac{1}{2}} \, d\rho$$

$$= \int_0^{2\pi} \frac{1+\cos 2\theta}{2} \, d\theta \cdot \lim_{\varepsilon \to 0^+} \left. \frac{\rho^{\frac{1}{2}}}{\frac{1}{2}} \right|_\varepsilon^1 =$$

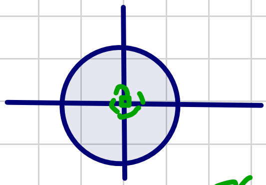
$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$= \left(\frac{1}{2} \int_0^{2\pi} d\theta + \frac{1}{2} \cdot \frac{1}{2} \int_0^{2\pi} \cos 2\theta \, 2 \, d\theta \right) \cdot \lim_{\varepsilon \to 0^+} 2 \cdot \left(1 - \underbrace{\sqrt{\varepsilon}}_{0} \right)$$

$$= \left(\frac{1}{\cancel{2}} \cdot \cancel{2}\pi + \frac{1}{4} \, \text{sen}\, 2\theta \Big|_0^{2\pi} \right) \cdot 2$$

$$= \left[\pi \cdot + \frac{1}{4}\cdot\left(\underbrace{\operatorname{sen}4\pi}_{0} - \underbrace{\operatorname{sen}0}_{0}\right)\right]\cdot 2 = 2\pi$$

d) $$\iint_{x^2+y^2\le 1} \ln\sqrt{x^2+y^2}\,dx\,dy = \lim_{\varepsilon\to 0^+}\int_{\theta=0}^{\theta=2\pi}\int_{\rho=\varepsilon}^{\rho=1} \ln\rho\cdot\rho\,d\rho\,d\theta$$



$$= \lim_{\varepsilon\to 0^+}\int_{\theta=0}^{\theta=2\pi} d\theta \cdot \underbrace{\int_{\rho=\varepsilon}^{\rho=1}\rho\cdot\ln\rho\,d\rho}_{\text{integrar por partes}} = \dots$$

$\nexists \ln 0$

já foi feito em um exercício na aula de hoje.

TERMINAR COMO EXERCÍCIO.