CÁLCULO 3, TURMA T3

13/05/26 – AULA 19

Encerramos a aula passada com exemplos do teste da derivada segunda para classificar os pontos críticos.

Combinando este teste com o teorema de Weierstrass(*), visto quando estudamos funções contínuas, obtemos o seguinte resultado:

COROLÁRIO: Se $f: \Omega \subset \mathbb{R}^n \to \mathbb{R}$ for uma função diferenciável em Ω, com Ω compacto do $\mathbb{R}^n$, então f possui pontos de máximo e mínimo, que ocorrerão nos pontos críticos ou na fronteira de Ω.

(*) T. WEIERSTRASS: Se $f: \Omega \subset \mathbb{R}^n \to \mathbb{R}$ for contínua em Ω, sendo Ω um compacto do $\mathbb{R}^n$, então f assume valores máximo e mínimo.

EXEMPLO: LISTA 06, EXERCÍCIO 06:

6. Encontre o máximo de $f(x,y) = 2x + y - 3xy$ no quadrado unitário $Q = [0,1] \times [0,1]$.
[Resp.: O máx. acontece na fronteira de Q e o valor é $f(1,0) = 2$].

Solução: Note que $Q = [0,1] \times [0,1]$ é compacto do $\mathbb{R}^2$ (limitado e fechado)

$$f : Q \subset \mathbb{R}^2 \to \mathbb{R}$$
$$f(x,y) = 2x + y - 3xy$$

Obter o máximo de f.

pontos críticos: onde $\nabla f = \vec{0}$.

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$$

$$\nabla f = (2 - 3y, 1 - 3x)$$

$$\nabla \vec{f} = (0,0) \iff (2-3y, 1-3x) = 0$$

$$\iff \begin{cases} 2 - 3y = 0 \to y = \frac{2}{3} \\ 1 - 3x = 0 \to x = \frac{1}{3} \end{cases}$$

PONTO CRÍTICO: $(x,y) = \left(\frac{1}{3}, \frac{2}{3} \right)$

Logo o máximo ocorrerá no ponto crítico $\left(\frac{1}{3}, \frac{2}{3}\right)$ ou na fronteira de $\Omega_2 = [0,1] \times [0,1]$.

max. ou min.?

$$H(x,y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(2-3y\right) = 0$$

$$f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(2-3y\right) = -3$$

$$f_{yx} = -3 \qquad \text{(T. de SCHWARZ)}$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(1-3x\right) = 0$$

$$\Rightarrow \quad H(x,y) = \begin{bmatrix} 0 & -3 \\ -3 & 0 \end{bmatrix}$$

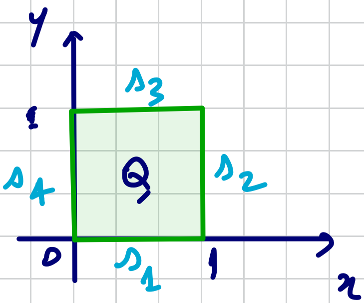
$$\Rightarrow \quad H\left(\frac{1}{3}, \frac{2}{3}\right) = \begin{bmatrix} 0 & -3 \\ -3 & 0 \end{bmatrix}$$

Logo;

$$\det\left(H\left(\tfrac{1}{3},\tfrac{2}{3}\right)\right) = \begin{vmatrix} 0 & -3 \\ -3 & 0 \end{vmatrix} = 0 - (+9) = -9 < 0$$

Logo, $\left(\frac{1}{3}, \frac{2}{3}\right)$ é um ponto de sela.

Portanto, o máx. def (e o mínimo também) vai ocorrer na fronteira de $Q = [0,1]\times[0,1]$.



Sejam s_1, s_2, s_3 e s_4 os lados de Q (fronteira).

Assim temos que; onde $f(x,y) = 2x+y-3xy$

• $P \in s_1 \iff f(a,0) : a \in [0,1]$

Nestes pontos:

$$f(P) = f(a,0) = 2a + 0 - 3a\cdot 0$$

$f(P) = 2a$ e o maior valor possível é quando $a=1$.

ou seja, $f(P) = 2.1 \Rightarrow f(P) = 2.$

isto é, $\boxed{f(1,0) = 2.}$ (*)

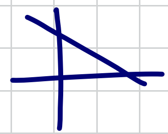
• $P \in s_2 \iff P(1,b)$; com $b \in [0,1]$

Neste caso,

$$f(P) = f(1,b) = 2.1 + b - 3.1.b$$

$$f(1,b) = 2 + b - 3b$$

$$f(1,b) = \underbrace{2 - 2b}_{\text{decresc.}}$$



O maior valor, neste caso, ocorre quando $b = 0$, ou seja,

$$f(1,b) = 2 - 2.0 = 2$$

$$\boxed{f(1,0) = 2} \quad (**)$$

• $P \in s_3 \iff P = (a,1)$; com $a \in [0,1]$.

Neste caso, teremos:

$$f(P) = f(a,1) = 2.a + 1 - 3.a.1$$

$$f(a,1) = 2a + 1 - 3a$$

$$f(a,1) = -a + 1$$

(decrescente)

Logo, o maior valor ocorre quando $a = 0$, ou seja; $f(0,1) = -0+1 \Rightarrow \boxed{f(0,1) = 1}$

(***) ,

• $P \in s_4 \Leftrightarrow P(0,b)$, com $b \in [0,1]$.

Neste caso, teremos:

$$f(P) = f(0,b) = 2.0 + b - 3.0.b$$

$$f(0,b) = b$$

(cres)

Logo, o maior valor ocorre quando $b=1$,

e será $\boxed{f(0,1) = 1}$ (****)

De (*), (**), (***) e (****) concluímos que o máximo ocorre na fronteira e será

$$f(1,0) = 2.$$

Deixaremos o restante da aula para a resolução de exercícios.

LISTA 06 :

8. Das funções de $\mathbb{R}^2 \to \mathbb{R}$ abaixo, classifique os extremos relativos, caso existam:

(a) $f(x,y) = x\ln(x+y)$.

a) $f(x,y) = x.\ln(x+y)$

pontos críticos: onde $\nabla f = \vec{0} = (0,0)$

$$\nabla f = \left(\frac{\partial f}{\partial x} , \frac{\partial f}{\partial y} \right) \quad , \text{ onde:}$$

$$\frac{\partial f}{\partial x} = u \cdot v = u \cdot v' + u' \cdot v$$

- $\frac{\partial f}{\partial x} = x \cdot \frac{1}{x+y} + 1 \cdot \ln(x+y)$

- $\frac{\partial f}{\partial y} = x \cdot \frac{1}{x+y} = \frac{x}{x+y}$

Assim: $\nabla f = \left(\frac{x}{x+y} + \ln(x+y) , \frac{x}{x+y} \right)$

$$\Rightarrow \nabla f = (0,0) \Leftrightarrow$$

$$\begin{cases} \frac{x}{x+y} + \ln(x+y) = 0 \\ \frac{x}{x+y} = 0 \Leftrightarrow x = 0 \end{cases}$$

Devemos observar que $x+y > 0$ (pelo "ln").

Como já temos $x = 0$, então $y > 0$.

Assim, a 1ª equação fica:

$$\frac{0}{y} + \ln(0+y) = 0 \quad , y > 0 ; \text{ ou seja:}$$

$\ln y = 0 \iff y = 1.$

Ou seja, temos um ponto crítico :

em $A(0,1)$

$$H(x,y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{x}{x+y} + \ln(x+y)\right)$$

$$\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}$$

$$f_{xx} = \frac{(x+y).1 - x.1}{(x+y)^2} + \frac{1}{x+y}$$

$$= \frac{y}{(x+y)^2} + \frac{1}{x+y}$$

$$f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{x}{x+y}\right) = \frac{(x+y).1 - x.1}{(x+y)^2}$$

$$= \frac{y}{(x+y)^2}$$

$f_{xy} = \dfrac{y}{(x+y)^2}$ (pelo T. de SCHWARZ)

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{x}{x+y}\right) = \frac{\partial}{\partial y}\Big(\underbrace{x}_{\text{CONST. PARA } y}\cdot(x+y)^{-1}\Big)$$

$$= x\cdot(-1)\cdot(x+y)^{-2}\cdot 1$$

$$\Rightarrow f_{yy} = \frac{-x}{(x+y)^2}$$

Assim:

$$H(x,y) = \begin{bmatrix} \frac{y}{(x+y)^2} + \frac{1}{x+y} & \frac{y}{(x+y)^2} \\ \frac{y}{(x+y)^2} & \frac{-x}{(x+y)^2} \end{bmatrix}$$

Dimo:

$$H(0,1) = \begin{bmatrix} \frac{1}{(1)^2} + \frac{1}{1} & \frac{1}{(0+1)^2} \\ \frac{1}{(0+1)^2} & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\Rightarrow \det\big(H(0,1)\big) = \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} = 0 - 1 = -1 < 0.$$

Logo $A(0,1)$ e' um ponto de sela.

Lista 05:

7. Se $u = f(x, y)$ é uma função diferenciável de x e y com $x = \rho\cos\theta$ e $y = \rho\sin\theta$, mostre que

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \rho}\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{\rho}$$

e

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \rho}\sin\theta + \frac{\partial u}{\partial \theta}\frac{\cos\theta}{\rho}$$

$$\begin{tikzcd}
 & & u = f(x,y) \arrow[dl] \arrow[dr] & & \\
 & x \arrow[dl] \arrow[d] & & y \arrow[d] \arrow[dr] & \\
\rho & \theta & & \rho & \theta
\end{tikzcd}$$

$$\left.\begin{aligned}
\frac{\partial u}{\partial \rho} &= \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \rho} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \rho} \\
\frac{\partial u}{\partial \theta} &= \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta}
\end{aligned}\right\} (*)$$

Temos:

$$x = \rho\cos\theta \quad \begin{cases} \dfrac{\partial x}{\partial \rho} = \cos\theta \\[2ex] \dfrac{\partial x}{\partial \theta} = -\rho\,\mathrm{sen}\,\theta. \end{cases}$$

$$y = \rho\,\mathrm{sen}\,\theta \quad \begin{cases} \dfrac{\partial y}{\partial \rho} = \mathrm{sen}\,\theta \\[2ex] \dfrac{\partial y}{\partial \theta} = \rho\cos\theta. \end{cases}$$

Substituindo em (*), vem:

$$(*)\begin{cases} \dfrac{\partial u}{\partial \rho} = \dfrac{\partial u}{\partial x} \cdot \cos\theta + \dfrac{\partial u}{\partial y} \cdot \operatorname{sen}\theta . \\[2ex] \dfrac{\partial u}{\partial \theta} = \dfrac{\partial u}{\partial x}(-\rho \operatorname{sen}\theta) + \dfrac{\partial u}{\partial y} \cdot \rho\cos\theta . \quad [\div \rho] \end{cases}$$

$$\begin{cases} \dfrac{\partial u}{\partial \rho} = \dfrac{\partial u}{\partial x} \cos\theta + \dfrac{\partial u}{\partial y} \operatorname{sen}\theta . \\[2ex] \dfrac{1}{\rho} \cdot \dfrac{\partial u}{\partial \theta} = -\dfrac{\partial u}{\partial x} \cdot \operatorname{sen}\theta + \dfrac{\partial u}{\partial y} \cos\theta . \end{cases}$$

$$\Downarrow$$

$$\dfrac{\partial u}{\partial y} \cdot \cos\theta = \dfrac{1}{\rho} \dfrac{\partial u}{\partial \theta} + \dfrac{\partial u}{\partial x} \cdot \operatorname{sen}\theta .$$

$$\boxed{\dfrac{\partial u}{\partial y} = \dfrac{1}{\rho\cos\theta} \cdot \dfrac{\partial u}{\partial \theta} + \dfrac{\partial u}{\partial x} \cdot \dfrac{\operatorname{sen}\theta}{\cos\theta}}$$

Levando para a 1.ª eq., fica:

$$\frac{\partial u}{\partial \rho} = \frac{\partial u}{\partial x} \cdot \cancel{\cos\theta} + \left[\frac{1}{\rho \cancel{\cos\theta}} \cdot \frac{\partial u}{\partial \theta} + \frac{\partial u}{\partial x} \cdot \frac{\operatorname{sen}\theta}{\cancel{\cos\theta}} \right] \cdot \operatorname{sen}\theta$$

$$\frac{\partial u}{\partial \rho} = \frac{\partial u}{\partial x} \cdot \cos\theta + \frac{1}{\rho} \frac{\operatorname{sen}\theta}{\cos\theta} \cdot \frac{\partial u}{\partial \theta} + \frac{\operatorname{sen}^2\theta}{\cos\theta} \cdot \frac{\partial u}{\partial x}$$

$$\frac{\partial u}{\partial \rho} = \frac{\partial u}{\partial x} \cdot \left[\cos\theta + \frac{\operatorname{sen}^2\theta}{\cos\theta} \right] + \frac{1}{\rho} \frac{\operatorname{sen}\theta}{\cos\theta} \cdot \frac{\partial u}{\partial \theta}.$$

$$\frac{\partial u}{\partial \rho} = \frac{\partial u}{\partial x} \cdot \frac{\overbrace{\cos^2\theta + \operatorname{sen}^2\theta}^{=1}}{\cos\theta} + \frac{1}{\rho} \cdot \frac{\operatorname{sen}\theta}{\cos\theta} \cdot \frac{\partial u}{\partial \theta}$$

$$\frac{1}{\cos\theta} \cdot \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \rho} - \frac{1}{\rho} \cdot \frac{\operatorname{sen}\theta}{\cos\theta} \cdot \frac{\partial u}{\partial \theta} \cdot (\times \cos\theta)$$

$$\frac{\partial u}{\partial x} = \cos\theta \cdot \frac{\partial u}{\partial \rho} - \cancel{\cos\theta} \cdot \frac{1}{\rho} \cdot \frac{\operatorname{sen}\theta}{\cancel{\cos\theta}} \frac{\partial u}{\partial \theta}$$

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \rho} \cdot \cos\theta - \frac{\operatorname{sen}\theta}{\rho} \cdot \frac{\partial u}{\partial \theta}}$$

Levando este resultado para a primeira igualdade em (*) obtemos a outra igualdade desejada:

$$\frac{\partial u}{\partial \rho} = \frac{\partial u}{\partial x} \cdot \cos\theta + \frac{\partial u}{\partial y} \cdot \operatorname{sen}\theta .$$

$$\frac{\partial u}{\partial \rho} = \left(\frac{\partial u}{\partial \rho} \cos\theta - \frac{\operatorname{sen}\theta}{\rho} \cdot \frac{\partial u}{\partial \theta} \right) \cdot \cos\theta + \underbrace{\frac{\partial u}{\partial y}}_{\text{isolar.}} \cdot \operatorname{sen}\theta$$

$$\frac{\partial u}{\partial \rho} = \frac{\partial u}{\partial \rho} \cdot \cos^2\theta - \frac{\operatorname{sen}\theta \cos\theta}{\rho} \cdot \frac{\partial u}{\partial \theta} + \frac{\partial u}{\partial y} \cdot \operatorname{sen}\theta .$$

$$\Rightarrow \frac{\partial u}{\partial y} \cdot \operatorname{sen}\theta = \frac{\partial u}{\partial \rho} - \cos^2\theta \cdot \frac{\partial u}{\partial \theta} + \frac{\operatorname{sen}\theta \cos\theta}{\rho} \cdot \frac{\partial u}{\partial \theta}$$

$$\frac{\partial u}{\partial y} \cdot \operatorname{sen}\theta = \underbrace{\left(1 - \cos^2\theta\right)}_{= \operatorname{sen}^2\theta} \cdot \frac{\partial u}{\partial \theta} + \frac{\operatorname{sen}\theta \cdot \cos\theta}{\rho} \cdot \frac{\partial u}{\partial \theta} .$$

$$\frac{\partial u}{\partial y} \cdot \operatorname{sen}\theta = \operatorname{sen}^2\theta \cdot \frac{\partial u}{\partial \theta} + \frac{\operatorname{sen}\theta \cdot \cos\theta}{\rho} \cdot \frac{\partial u}{\partial \theta} \quad \left(\div \operatorname{sen}\theta\right)$$

$$\boxed{\frac{\partial u}{\partial y} = \operatorname{sen}\theta \cdot \frac{\partial u}{\partial \theta} + \frac{\cos\theta}{\rho} \cdot \frac{\partial u}{\partial \theta}}$$