CÁLCULO 3, TURMA T3

04/05/26 – AULA 15

REGRA DA CADEIA PARA FUNÇÕES DE $\mathbb{R}^m$ EM $\mathbb{R}^n$.

TEOREMA (REGRA DA CADEIA) Sejam $f: \mathbb{R}^m \to \mathbb{R}^n$ uma função diferenciável em $a \in \mathbb{R}^m$, $g: \mathbb{R}^n \to \mathbb{R}^k$ uma função diferenciável em $b = f(a) \in \mathbb{R}^n$. Então

$g \circ f: \mathbb{R}^m \to \mathbb{R}^k$ é diferenciável em $a \in \mathbb{R}^m$,

com $$(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$$

Em notação de diferenciais:

$$d_a(g \circ f) = d_{f(a)}g \cdot d_a f$$

PRODUTO MATRICIAL

A demonstração deste teorema é omitida por usar recursos que fogem de um curso tradicional de Cálculo.

Vejamos um exemplo de aplicação.

Ex: Dada $f: \mathbb{R}^3 \to \mathbb{R}^2$, $f(x,y,z) = (x^2+z^2, x.y.z)$

e $g: \mathbb{R}^2 \to \mathbb{R}^4$, $g(x,y) = (x+y, x^2-y^2, x, xy)$

Determine $(g \circ f)'(a)$; onde $a = (1,2,1)$.

Solução: Uma das formas seria montar a composição $(g \circ f)(x,y,z)$ e, depois, calcular $d_a(g \circ f)$ [a matriz jacobiana]

No entanto, o que faremos aqui será usar a regra da cadeia:

$$(g \circ f)'(a) = g'(f(a)) \cdot f'(a) \, ;$$

ou seja:

$$\underset{4\times 3}{\left[d_a (g \circ f) \right]} = \underset{4\times 2}{\left[d_{f(a)} g \right]} \cdot \underset{2\times 3}{\left[d_a f \right]}$$

PRODUTO MATRICIAL.

Note que $\left.\begin{array}{l} f: \mathbb{R}^3 \to \mathbb{R}^2 \\ g: \mathbb{R}^2 \to \mathbb{R}^4 \end{array}\right\} \Rightarrow g \circ f: \mathbb{R}^3 \to \mathbb{R}^4$.

Assim, teremos: $\left[d_a\, g \circ f\right]_{4\times 3}$; $\left[d_a f\right]_{2\times 3}$ e $\left[d_{f(a)}\, g\right]_{4\times 2}$

Sendo $f(x,y,z) = (\underbrace{x^2+z^2}_{f_1}, \underbrace{xyz}_{f_2})$;

$$g(x,y) = (\underbrace{x+y}_{g_1}, \underbrace{x^2-y^2}_{g_2}, \underbrace{x}_{g_3}, \underbrace{xy}_{g_4})$$

$$\left[d_a\, g \circ f\right]_{4\times 3} = \left[d_{f(a)}\, g\right]_{4\times 2} \cdot \left[d_a f\right]_{2\times 3}$$

$$= \begin{bmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \\ \frac{\partial g_3}{\partial x} & \frac{\partial g_3}{\partial y} \\ \frac{\partial g_4}{\partial x} & \frac{\partial g_4}{\partial y} \end{bmatrix}_{f(a)} \cdot \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{bmatrix}_a =$$

$$= \begin{bmatrix} 1 & 1 \\ 2x & -2y \\ 1 & 0 \\ y & x \end{bmatrix}_{f(a)} \cdot \begin{bmatrix} 2x & 0 & 2z \\ yz & xz & xy \end{bmatrix}_{a}$$

Como $a = (1,2,1)$, então $f(a) = (1^2+1^2, 1.2.1)$

$$\Rightarrow f(a) = (2,2).$$

Assim:

$$= \begin{bmatrix} 1 & 1 \\ 2x & -2y \\ 1 & 0 \\ y & x \end{bmatrix}_{\underset{x\ \ y}{(2,2)}} \cdot \begin{bmatrix} 2x & 0 & 2z \\ yz & xz & xy \end{bmatrix}_{\underset{x\ y\ z}{(1,2,1)}} =$$

$$= \begin{bmatrix} 1 & 1 \\ 2.2 & -2.2 \\ 1 & 0 \\ 2 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2.1 & 0 & 2.1 \\ 2.1 & 1.1 & 1.2 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 1 \\ 4 & -4 \\ 1 & 0 \\ 2 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & 2 \\ 2 & 1 & 2 \end{bmatrix} =$$

$$= \begin{bmatrix} 1.2 + 1.2 & 1.0 + 1.1 & 1.2 + 1.2 \\ 4.2 - 4.2 & 4.0 - 4.1 & 4.2 - 4.2 \\ 1.2 + 0.2 & 1.0 + 0.1 & 1.2 + 0.2 \\ 2.2 + 2.2 & 2.0 + 2.1 & 2.2 + 2.2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 1 & 4 \\ 0 & -4 & 0 \\ 2 & 0 & 2 \\ 8 & 2 & 8 \end{bmatrix} = d_a(g \circ f)$$

Quando a função for escalar, o teorema da regra da Cadeia possui um enunciado (e uma demonstração) mais simples, como veremos a seguir:

TEOREMA: (REGRA DA CADEIA PARA FUNÇÕES ESCALARES)

Seja $u = f(x,y)$ uma função diferenciável, tal que $x = x(r,s)$ e $y = y(r,s)$ diferenciáveis, tais que $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial x}{\partial r}$, $\frac{\partial y}{\partial r}$, $\frac{\partial x}{\partial s}$ e $\frac{\partial y}{\partial s}$ existam. Então:

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r}$$

e

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s} .$$

DEMONSTR.: Seja $u = f(x,y)$ nas hipóteses do teorema. Assim, sendo u diferenciável, então:

$$\boxed{\Delta u = \frac{\partial u}{\partial x} \cdot \Delta x + \frac{\partial u}{\partial y} \cdot \Delta y + \varepsilon_1 \cdot \Delta x + \varepsilon_2 \cdot \Delta y} \quad (*),$$

onde $\varepsilon_1, \varepsilon_2 \underset{(\Delta x, \Delta y) \to 0}{\longrightarrow} 0$.

Vamos determinar $\frac{\partial u}{\partial r}$. Neste caso, a variável s ficará constante. Então; tem-se os incrementos:

$$\Delta x = x(r+\Delta r, s) - x(r,s)$$
$$\Delta y = y(r+\Delta r, s) - y(r,s)$$

Dividindo (*) por $\Delta r \neq 0$, obtemos:

$$\frac{\Delta u}{\Delta r} = \frac{\partial u}{\partial x} \cdot \frac{\Delta x}{\Delta r} + \frac{\partial u}{\partial y} \cdot \frac{\Delta y}{\Delta r} + \varepsilon_1 \cdot \frac{\Delta x}{\Delta r} + \varepsilon_2 \cdot \frac{\Delta y}{\Delta r}$$

$$\Rightarrow \frac{\Delta u}{\Delta r} = \frac{\partial u}{\partial x} \cdot \frac{x(r+\Delta r, s) - x(r,s)}{\Delta r} +$$

$$+ \frac{\partial u}{\partial y} \cdot \frac{y(r+\Delta r, s) - y(r,s)}{\Delta r} +$$

$$+ \varepsilon_1 \cdot \frac{x(r+\Delta r, s) - x(r,s)}{\Delta r} + \varepsilon_2 \cdot \frac{y(r+\Delta r, s) - y(r,s)}{\Delta r}$$

Tomando o limite com Δr, obtemos:

$$\underbrace{\lim_{\Delta r \to 0} \frac{\Delta u}{\Delta r}}_{\frac{\partial u}{\partial r}} = \frac{\partial u}{\partial x} \cdot \underbrace{\lim_{\Delta r \to 0} \frac{x(r+\Delta r, s) - x(r,s)}{\Delta r}}_{\frac{\partial x}{\partial r}} +$$

$$+ \frac{\partial u}{\partial y} \cdot \underbrace{\lim_{\Delta r \to 0} \frac{y(r+\Delta r, s) - y(r,s)}{\Delta r}}_{\frac{\partial y}{\partial r}} +$$

$$+ \underbrace{\lim_{\Delta r \to 0} \varepsilon_1}_{0} \cdot \underbrace{\frac{x(r+\Delta r, s) - x(r,s)}{\Delta r}}_{\frac{\partial x}{\partial x}} +$$

$$+ \underbrace{\lim_{\Delta r \to 0} \varepsilon_2}_{0} \cdot \underbrace{\frac{y(r+\Delta r, s) - y(r,s)}{\Delta r}}_{\frac{\partial y}{\partial r}}$$

$$\Rightarrow \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r} + 0 + 0$$

$$\Longrightarrow \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r} .$$

Analogamente prova-se a outra igualdade.

□

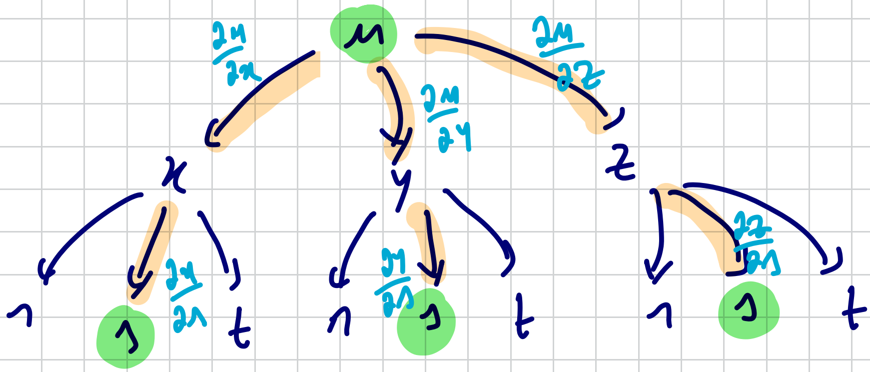
Obs.: Este resultado pode ser extendido para funções escalares a mais variáveis.

Ex.: $u = u(x, y, z)$; onde

$$\begin{cases} x = x(r, s, t) \\ y = y(r, s, t) \\ z = z(r, s, t) \end{cases}$$

Então; o que seria $\dfrac{\partial u}{\partial t} = ?$ $\dfrac{\partial u}{\partial s} = ?$ $\dfrac{\partial u}{\partial r} = ?$

Para obter a fórmula, fazemos um esquema em "árvore" como segue:



Assim:

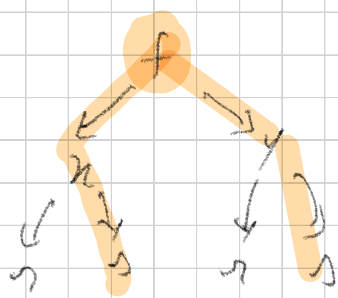
$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial s}$$

E assim para as demais derivadas.

Ex.: $f(x,y) = x^2 + \ln xy$; com

$$x = r^2 + s^2 \quad \text{e} \quad y = \frac{r}{s} = r.s^{-1}$$

Obtenha $\frac{\partial f}{\partial r}$ e $\frac{\partial f}{\partial s}$.



Solução:

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r} \;;$$

onde:

- $\frac{\partial f}{\partial x} = 2x + \frac{\cancel{y}}{x\cancel{y}} \Rightarrow \frac{\partial f}{\partial x} = 2x + \frac{1}{x}$

- $\frac{\partial x}{\partial r} = 2r$

- $\frac{\partial f}{\partial y} = \frac{\cancel{x}}{\cancel{x}y} = \frac{1}{y} \Rightarrow \frac{\partial f}{\partial y} = \frac{1}{y}$

• $\dfrac{\partial y}{\partial r} = 1 \cdot s^{-1} = \dfrac{1}{s}$.

Portanto, obtemos:

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$= \left(2\underbrace{x}_{r^2+s^2} + \frac{1}{\underbrace{x}}\right) \cdot 2r + \frac{1}{y} \cdot \frac{1}{s}$$

$$= \left(2(r^2+s^2) + \frac{1}{r^2+s^2}\right) \cdot 2r + \frac{1}{\frac{r}{s}} \cdot \frac{1}{s}$$

$$= \left(2r^2+2s^2 + \frac{1}{r^2+s^2}\right) 2r + \frac{\cancel{s}}{r} \cdot \frac{1}{\cancel{s}}$$

$$= 4r^3 + 4rs^2 + \frac{2r}{r^2+s^2} + \frac{1}{r}$$

Analogamente faz-se $\dfrac{\partial f}{\partial s}$:

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s} ; \quad \text{onde:}$$

$$\frac{\partial f}{\partial x} = 2x + \frac{1}{x} ; \quad \frac{\partial f}{\partial y} = \frac{1}{y} ;$$

$$\frac{\partial x}{\partial s} = 2s ; \quad \frac{\partial y}{\partial s} = -1 \cdot r \cdot s^{-2} = -\frac{r}{s^2}$$

Assim:

$$\frac{\partial f}{\partial s} = \left(2x + \frac{1}{x}\right) \cdot 2s + \frac{1}{y} \cdot \left(-\frac{r}{s^2}\right) = \ldots$$