CÁLCULO 1 A

18/12/25 – AULA 23

Iniciamos na aula passada o estudo de integrais. Apresentamos uma motivação do conceito de integral definida e, em seguida, vimos a integral indefinida, como o processo inverso à derivação.

No que segue, veremos as principais fórmulas.

Sendo u, v funções de x; então:

01) $\int k.dx = ?$

Ou seja, qual é a função cuja derivada seja uma constante k?

Resposta: $\frac{d}{dx}(k.x + C) = k$.

Portanto, $\int k\,dx = k.x + C$.

Ex.: $\int 3dx = ?$

$$\int 3\,dx = 3x + C, \text{ pois}$$

$$(3x+c)' = 3$$

02) $$\int (u+v)\,dx = \int u\,dx + \int v\,dx.$$

De fato;

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$$

03) $$\int v^{k} \cdot dv = \frac{v^{k+1}}{k+1} + C$$

Ex.: $$\int x^4\,dx = ?$$

$$v = x \Rightarrow \frac{dv}{dx} = 1$$

$$\Rightarrow dv = dx$$

$$k = 4$$

Neste caso,

$$\int x^4 \cdot dx = \frac{x^{4+1}}{4+1} + C = \frac{x^5}{5} + C$$

Combinando estes resultados, podemos, por exemplo, resolver:

$$\int (x^3 - 3x^2 + 2)\,dx = ?$$

$$\int (x^3 - 3x^2 + 2)\,dx = \int x^3\,dx - \int 3x^2\,dx + \int 2\,dx$$

$\int u^k\,du$ $\qquad$ $\int k\,du$

$$= \frac{x^4}{4} - \cancel{3} \cdot \frac{x^3}{\cancel{3}} + 2x + C$$

$$= \frac{x^4}{4} - x^3 + 2x + C$$

De fato: $\frac{d}{dx}\left(\frac{x^4}{4} - x^3 + 2x + C\right) =$

$$= \frac{\cancel{4}x^3}{\cancel{4}} - 3x^2 + 2 + 0 =$$

$$= x^3 - 3x^2 + 2$$

04) $\int k \cdot v \, dx = k \cdot \int v \, dx.$

uma constante multiplicando pode ir para fora do símbolo de integral.

05) $\int \frac{dv}{v} = \ln |v| + c.$

Ex.: a) $\int \frac{dx}{x+1} = ?$

$\int \frac{dv}{v}.$

$v = x+1$

$\frac{dv}{dx} = 1.$

$dv = 1 \, dx$

$$\int \frac{dx}{x+1} = \int \frac{dv}{v} = \ln |v| + c$$

$$= \ln |x+1| + c.$$

b) $\displaystyle\int \frac{dx}{3x-2} = ?$ $\qquad\qquad \displaystyle\int \frac{du}{u}$

$u = 3x - 2 \Longrightarrow \dfrac{du}{dx} = 3$

$du = 3 \cdot dx.$

$dx = \dfrac{du}{3}$

Assim: $\displaystyle\int \frac{dx}{3x-2} = \int \frac{\frac{du}{3}}{u} = \frac{1}{3}\int \frac{du}{u}$

$= \dfrac{1}{3} \cdot \ln|u| + C$

$= \dfrac{1}{3} \cdot \ln|3x-2| + C$

Ob) $\displaystyle\int e^u du = e^u + c.$;

pois $\dfrac{d}{dx}(e^u) = e^u \cdot u' = e^u \cdot \dfrac{du}{dx}$

$\Longrightarrow \displaystyle\int e^u du = e^u + c.$

Ex: a) $\int e^{3x-5}\, dx = ?$

$u = 3x-5 \Rightarrow \frac{du}{dx} = 3$

$\Rightarrow du = 3dx$

$\Rightarrow dx = \frac{du}{3}$

Dizno:

$$\int e^{3x-5}\, dx = \int e^{u} \cdot \frac{du}{3} = \frac{1}{3} \cdot \int e^{u} du$$

$$= \frac{1}{3} e^{u} + c = \frac{1}{3} \cdot e^{3x-5} + c.$$

b) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} \cdot dx = ?$

$\int e^{u} du = e^{u} + c.$

$u = \sqrt{x} = x^{\frac{1}{2}}$

$\Rightarrow \frac{du}{dx} = \frac{1}{2} x^{-\frac{1}{2}} \Rightarrow du = \frac{1}{2} \frac{1}{\sqrt{x}} dx$

$\Rightarrow 2 \cdot du = \frac{dx}{\sqrt{x}}$

Assim, vamos obter:

$$\int \frac{e^{\sqrt{x}}\,dx}{\sqrt{x}} = \int e^{v} \cdot \underbrace{2dv} = 2 \cdot \int e^{v} dv$$

$\sqrt{x} = v$, $\frac{dx}{\sqrt{x}} = 2dv$

$$= 2 \cdot e^{v} + C$$

$$= 2 \cdot e^{\sqrt{x}} + C.$$

De fato, se derivarmos a resposta, vamos obter:

$$\frac{d}{dx}\left(2 \cdot e^{\sqrt{x}} + C\right) = \cancel{2} \cdot e^{\sqrt{x}} \cdot \frac{1}{\cancel{2}} x^{-\frac{1}{2}} + 0$$

$$= \frac{e^{\sqrt{x}}}{\sqrt{x}}$$

07) $\int \operatorname{sen} v \, dv = ?$

Ou seja, qual é a função cuja derivada é seno.

$$\int \operatorname{sen} v \, dv = -\cos v + C.$$

De fato: $\frac{d}{dv}(-\cos v + c) = -(-\operatorname{sen} v)$

$= \operatorname{sen} v.$

Ex.: $\int \operatorname{sen}(5x-7)\,dx = ?$

$\int \operatorname{sen} v \, dv$

$v = 5x - 7$

$\frac{dv}{dx} = 5 \Rightarrow dv = 5\,dx$

$dx = \frac{dv}{5}$

Disso:

$$\int \operatorname{sen}(5x-7)\,dx = \int \operatorname{sen} v \cdot \frac{dv}{5} =$$

$$= \frac{1}{5} \cdot \int \operatorname{sen} v \, dv = \frac{1}{5} \cdot (-\cos v + c)$$

$$= -\frac{1}{5} \cos v + c$$

$$= -\frac{1}{5} \cdot \cos(5x-7) + C$$

08) $\int \cos u \, du = \operatorname{sen} u + C.$

Ex: $\int (4x-2) \cdot \cos(x^2 - x) \cdot dx = ?$

$$\int \cos u \, du = \operatorname{sen} u + C.$$

$$u = x^2 - x \Rightarrow \frac{du}{dx} = 2x - 1$$

$$\Rightarrow du = (2x-1)dx$$

Assim:

$$\int (4x-2) \cdot \cos(x^2 - x)dx =$$

$$\int 2 \underbrace{(2x-1) \cdot \cos(\underbrace{x^2-x}_{u})dx}_{du} =$$

$$= 2 \cdot \int \cos u \cdot du = 2 \cdot \operatorname{sen} u + C$$

$$= 2 \cdot \operatorname{sen}(x^2 - x) + C$$

09) $\int \sec^2 u \, du = \tan u + C$

Isto porque $\dfrac{d}{du}(\tan u + C) = \sec^2 u$.

<u>Ex.</u>: $\int \sec^2(1-2x)\,dx = ?$

$\int \sec^2 u \, du$

$u = 1-2x$

$\Rightarrow \dfrac{du}{dx} = -2$

$du = -2dx$

$dx = -\frac{1}{2} du$

Assim, vamos obter:

$$\int \sec^2(1-2x)\,dx = \int \sec^2 u \cdot \left(-\frac{1}{2} du\right) =$$

$$= -\frac{1}{2} \int \sec^2 u \cdot du =$$

$$= -\frac{1}{2} \tan u + C =$$

$$= -\frac{1}{2} \tan(1-2x) + C$$

10) $\int \sec v \cdot \tan v \, dv = \sec v + C$, pois

$$\frac{d}{dv}(\sec v + c) = \sec v \cdot \tan v$$

Ex.: $\displaystyle\int \frac{\sec\sqrt{x} \cdot \tan\sqrt{x}}{\sqrt{x}} \, dx = ?$

$$v = \sqrt{x} = x^{\frac{1}{2}} \Rightarrow \frac{dv}{dx} = \frac{1}{2} x^{-\frac{1}{2}}$$

$$\Rightarrow dv = \frac{1}{2\sqrt{x}} dx$$

$$\Rightarrow \frac{dx}{\sqrt{x}} = 2 \cdot dv$$

Com isso, concluímos que:

$$\int \frac{\sec \overset{v}{\sqrt{x}} \cdot \tan \overset{v}{\sqrt{x}}}{\underbrace{\sqrt{x}}_{2\,dv}} \, dx = \int \sec v \cdot \tan v \cdot (2\,dv)$$

$$= 2 \cdot \int \sec v \cdot \tan v \, dv =$$

$$= 2 \cdot \sec v + C$$

$$= 2 \sec\sqrt{x} + C$$

11) $\int \tan x \, dx = ?$

$$\int \tan x \, dx = \int \frac{\operatorname{sen} x}{\cos x} dx = -\int \frac{du}{u} \overset{*}{=}$$

onde $u = \cos x$

$\Rightarrow du = -\operatorname{sen} x \, dx$

$\Rightarrow \operatorname{sen} x \, dx = -du$

$$\overset{*}{=} -\ln|u| + c = -\ln|\cos x| + c$$

$$= \ln|\cos x|^{-1} + c$$

$$= \ln\left|\frac{1}{\cos x}\right| + c \Rightarrow \ln|\sec x| + c.$$

conclusão: $\int \tan x \, dx = \ln|\sec x| + c.$

EXERCÍCIOS: Calcule as integrais:

a) $\int (\sqrt{x} - 3x^2) dx.$

c) $\int \cos(7x) dx.$

b) $\int \frac{dx}{3-2x}.$

d) $\int e^{4x-2} dx.$

solução:

(a) $\int (\sqrt{x} - 3x^2)\,dx = \int \sqrt{x}\,dx - 3\cdot\int x^2\,dx$

$$= \int x^{\frac{1}{2}}\,dx - 3\cdot\int x^2\,dx =$$

$$\int u^k\,du = \frac{u^{k+1}}{k+1} + c$$

$$u = x \Rightarrow du = dx$$

$$= \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \cancel{3}\cdot\frac{x^3}{\cancel{3}} + c = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - x^3 + c$$

$$= \frac{2}{3}\cdot x^{\frac{3}{2}} - x^3 + c$$

b) $\int \frac{dx}{3-2x} = \int \frac{-\frac{du}{2}}{u} = -\frac{1}{2}\cdot\int \frac{du}{u} =$

$\int \frac{du}{u}$

$u = 3-2x$

$du = -2\,dx \Rightarrow dx = -\frac{du}{2}$

$\Leftrightarrow -\frac{1}{2} \ln|u| + c = -\frac{1}{2} \cdot \ln|3-2x| + c$

c) $\int \cos(7x)\,dx$.

$\int \cos u \, du = \operatorname{sen} u + c$

$u = 7x$

$\Rightarrow \frac{du}{dx} = 7 \Rightarrow du = 7dx$

$\Rightarrow dx = \frac{du}{7}$

$\Rightarrow \int \cos(7x)dx = \int \cos u \cdot \frac{du}{7} =$

$= \frac{1}{7} \int \cos u + c = \frac{1}{7} \cdot \operatorname{sen} u + c$

$= \frac{1}{7} \cdot \operatorname{sen}(7x) + c$

d) $\int e^{4x-2}\,dx = ?$

$\int e^u du = e^u + c$

$u = 4x-2 \Rightarrow \frac{du}{dx} = 4 \Rightarrow du = 4dx$

$dx = \frac{du}{4}$

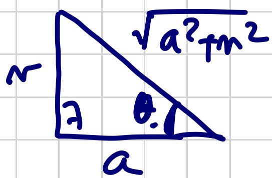
Assim, vamos obter:

$$\int e^{\overbrace{4x-2}^{v}}\underbrace{dx}_{\frac{dv}{4}} = \int e^{v} \cdot \frac{dv}{4} = \frac{1}{4}\int e^{v}\,dv =$$

$$= \frac{1}{4} \cdot e^{v} + c = \underline{\frac{1}{4} \cdot e^{4x-2} + c}$$

12): $\displaystyle\int \frac{dv}{v^2+a^2} = \frac{1}{a} \cdot \arctan\left(\frac{v}{a}\right) + c \quad , a \in \mathbb{R}$

A ideia consiste em fazer uma substituição adequada de variável. pense v^2+a^2 como soma de quadrados de um cateto de um triângulo retângulo de lados a e v.



Da trigonometria; $\tan\theta = \dfrac{v}{a}$

$$\Rightarrow v = a \cdot \tan\theta.$$

Derivando em θ, vem:

$$\frac{dx}{d\theta} = a \cdot \sec^2\theta$$

$$\Rightarrow \boxed{dx = a \cdot \sec^2\theta \, d\theta.}$$

Além disso;

$$\cos\theta = \frac{a}{\sqrt{x^2+a^2}} \Rightarrow \sqrt{x^2+a^2} = \frac{a}{\cos\theta}$$

$$\Rightarrow \left(\sqrt{x^2+a^2}\right)^2 = \left(a \cdot \sec\theta.\right)^2$$

$$\boxed{x^2+a^2 = a^2 \cdot \sec^2\theta.}$$

Assim:

$$\int \frac{dx}{x^2+a^2} = \int \frac{\cancel{a} \cdot \cancel{\sec^2\theta} \cdot d\theta}{a^{\cancel{2}} \cancel{\sec^2\theta}} = \int \frac{1}{a} \cdot d\theta =$$

$$= \frac{1}{a} \cdot \int d\theta = \frac{1}{a} \cdot \theta + C$$

Como $\tan\theta = \frac{x}{a}$, teremos que

$\theta = \arctan\left(\frac{x}{a}\right)$, e disso,

segue que:

$$\int \frac{dv}{v^2+a^2} = \frac{1}{a}\cdot\theta + C = \frac{1}{a}\cdot \arctan\left(\frac{v}{a}\right) + c.$$

Ex.: $\int \frac{dx}{x^2+4} = ?$

$$\int \frac{dv}{v^2+a^2}$$

$v = x \qquad a^2 = 4 \Rightarrow a = 2$

$$\Rightarrow \int \frac{dx}{x^2+4} = \frac{1}{2}\cdot \arctan\left(\frac{x}{2}\right) + c.$$

Obs: por quê $\int \frac{dv}{v}$ não funciona neste caso?

$$\int \frac{dx}{x^2+4} \qquad \int \frac{dv}{v}$$

$$v = x^2+4 \rightarrow \frac{dv}{dx} = 2x$$

$$\Rightarrow dx = \frac{dv}{2x} \rightarrow \text{este } x \text{ atrapalha}$$