CÁLCULO 1A

06/11/25 - AULA 14

Na aula passada estudamos as primeiras regras de derivação. Vimos que:

01) $y = k \Rightarrow y' = 0$

02) $y = k \cdot x \Rightarrow y' = k$

03) $y = u + v \Rightarrow y' = u' + v'$

04) $y = v^k \Rightarrow y' = k \cdot v^{k-1} \cdot v'$

EXEMPLOS:

(a) $y = x^5 \Rightarrow y' = ?$

$$y = 5x^4$$

(b) $y = \underbrace{x^3}_{u} + \underbrace{4x}_{v} \Rightarrow y' = ?$

$y' = u' + v'$

$u = x^3 \Rightarrow u' = 3x^2$

$v = 4x \Rightarrow v' = 4$

$$\Rightarrow \underset{\sim}{y'} = u' + v' = \underset{\sim\sim\sim}{3x^2+4}$$

(c) $y = 5x^3 - 2x^2 + 4x - 5$. $y' = ?$

$y' = 5.(3x^2) - 2.(2x^1) + 4.1 - 0$

$y' = 15x^2 - 4x + 4$.

05) $y = k.v \Rightarrow y' = k.v'$

De fato:

$f(x) = k.v(x)$. Então:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} =$$

$$= \lim_{h\to 0} \frac{k.v(x+h) - k.v(x)}{h} =$$

$$= \lim_{h\to 0} k.\left(\frac{v(x+h) - v(x)}{h}\right) =$$

$$= k \cdot \underbrace{\lim_{h \to 0} \frac{v(x+h) - v(x)}{h}}_{v'(x)} = k \cdot v'(x).$$

06) $y = u \cdot v \Rightarrow y' = u \cdot v' + u' \cdot v.$

Seja $f(x) = u(x)\, v(x)$; então:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} =$$

$$= \lim_{h \to 0} \frac{u(x+h) \cdot v(x+h) - u(x) \cdot v(x)}{h} =$$

$$= \lim_{h \to 0} \frac{1}{h} \cdot \left[u(x+h) \cdot v(x+h) - u(x) \cdot v(x+h) + u(x) \cdot v(x+h) - u(x) \cdot v(x) \right]$$

$$= \lim_{h \to 0} \underbrace{\frac{1}{h} \left[u(x+h) - u(x) \right]}_{u'(x)} \cdot \underbrace{v(x+h)}_{\to v(x)} + \underbrace{\frac{1}{h} \left[v(x+h) - v(x) \right]}_{v'(x)} u(x)$$

$$= u'(x) \cdot v(x) + v'(x) \cdot u(x)$$

CONCLUSÃO:

$$y = u \cdot v \implies y' = u \cdot v' + u' \cdot v$$

EXEMPLOS:

01) $y = (x^2 + x^3 - 1) \cdot (3x^4 - x^5)$. $y' = ?$

$$u = x^2 + x^3 - 1 \implies u' = 2x + 3x^2$$

$$v = 3x^4 - x^5 \implies v' = 12x^3 - 5x^4$$

Assim, obtemos:

$$y' = u \cdot v' + u' \cdot v =$$

$$= (x^2 + x^3 - 1) \cdot (12x^3 - 5x^4) + (2x + 3x^2) \cdot (3x^4 - x^5)$$

07) $y = \dfrac{u}{v} \Rightarrow y' = \dfrac{v \cdot u' - u \cdot v'}{v^2}$

De fato, seja $f(x) = \dfrac{u(x)}{v(x)}$. Então:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\dfrac{u(x+h)}{v(x+h)} - \dfrac{u(x)}{v(x)}}{h}$$

$$= \lim_{h \to 0} \frac{\dfrac{u(x+h) \cdot v(x) - u(x) \cdot v(x+h)}{v(x+h) \cdot v(x)}}{h} =$$

$$= \lim_{h \to 0} \frac{1}{h} \cdot \frac{u(x+h) \cdot v(x) - u(x) \cdot v(x) + u(x) \cdot v(x) - u(x) \cdot v(x+h)}{v(x+h) \cdot v(x)}$$

$$= \lim_{h \to 0} \frac{v(x) \cdot \left[\dfrac{u(x+h) - u(x)}{h}\right] - u(x) \cdot \left[\dfrac{v(x+h) - v(x)}{h}\right]}{v(x+h) \cdot v(x)}$$

$u'(x)$ $v'(x)$

$v(x)$

$$= \frac{v(x)\cdot u'(x) - u(x)\, v'(x)}{v(x)\, v(x)} = \frac{v\cdot u' - u\cdot v'}{v^2}$$

<u>Ex.:</u> $y = \dfrac{3x+1}{1-2x} \Rightarrow y' = ?$

$$y = \frac{u}{v} \Rightarrow y' = \frac{v\, u' - u\cdot v'}{v^2} ;$$

onde:

$$\begin{cases} u = 3x+1 \Rightarrow u' = 3 \\ v = 1-2x \Rightarrow v' = -2 \end{cases}$$

$$\Rightarrow y' = \frac{v\cdot u' - u\cdot v'}{v^2} = \frac{(1-2x)\cdot 3 - (3x+1)(-2)}{(1-2x)^2}$$

$$y' = \frac{3 - \cancel{6x} + \cancel{6x} + 2}{(1-2x)^2} = \underline{\frac{5}{(1-2x)^2}}$$

07) $y = \ln v \Longrightarrow y' = \dfrac{v'}{v}$

$$\boxed{\ln v = \log_e v}$$

EXEMPLOS: (a) $y = \ln(3x - x^3) \Longrightarrow y' = ?$

$$y = \ln v \Longrightarrow y' = \frac{v'}{v}$$

$$v = 3x - x^3 \Longrightarrow v' = 3 - 3x^2$$

$$\Longrightarrow y' = \frac{v'}{v} = \frac{3 - 3x^2}{3x - x^3}$$

(b) $y = x \cdot \ln(1-x)$. $y' = ?$

$$y = u \cdot v \Longrightarrow y' = u \cdot v' + u' \cdot v$$

$$\begin{cases} u = x \longrightarrow u' = 1 \\ v = \ln(1-x) \longrightarrow v' = \dfrac{(1-x)'}{1-x} = \dfrac{-1}{1-x} \end{cases}$$

Assim, obtemos:

$$y' = u \cdot v' + u' \cdot v$$

$$y' = x \cdot \left(-\frac{1}{1-x}\right) + 1 \cdot \ln(1-x)$$

$$y' = -\frac{x}{1-x} + \ln(1-x)$$

(c) $\quad y = \ln(\ln(3x-1))$

$$y = \ln v$$

$$v = \ln(3x-1)$$

$$\longrightarrow v' = \frac{(3x-1)'}{3x-1} = \frac{3}{3x-1}$$

$$v' = (\ln(3x-1))'$$

Assim:

$$y' = \frac{(\ln(3x-1))'}{\ln(3x-1)} = \frac{\frac{3}{3x-1}}{\ln(3x-1)} =$$

$$= \frac{3}{3x-1} \cdot \frac{1}{\ln(3x-1)} = \frac{3}{(3x-1) \cdot \ln(3x-1)}$$

09) $y = e^{u} \Rightarrow y' = e^{u} \cdot u'$.

Ex: $y = e^{3x+7} \Rightarrow y' = ?$

$u = 3x+7 \Rightarrow u' = 3$

$\Rightarrow y' = e^{u} \cdot u'$

$y' = e^{3x+7} \cdot 3$

$\Rightarrow y' = 3 \cdot e^{3x+7}$

10) $y = \operatorname{sen} u \Rightarrow y' = \cos u \cdot u'$

Ex: $y = \operatorname{sen}(x^3)$. $y' = ?$

$u = x^3 \Rightarrow u' = 3x^2$

$\Rightarrow y' = \cos u \cdot u' = \cos(x^3) \cdot 3x^2$

$\Rightarrow y' = 3x^2 \cdot \cos(x^3)$.

11) $y = \cos v \Rightarrow y' = -\operatorname{sen} v \cdot v'$

Ex.: $y = \cos(x - x^2) \Rightarrow y' = ?$

$v = x - x^2 \Rightarrow v' = 1 - 2x$

$\Rightarrow y' = -\operatorname{sen} v \cdot v'$

$y' = -\operatorname{sen}(x - x^2) \cdot (1 - 2x)$

$y' = -(1 - 2x) \cdot \operatorname{sen}(x - x^2)$.

$y' = (2x - 1) \cdot \operatorname{sen}(x - x^2)$

12) $y = \tan v \Rightarrow y' = \sec^2 v \cdot v'$

De fato:

$$y = \tan v = \frac{\operatorname{sen} v}{\cos v} = \frac{u}{w}$$

$$y' = \frac{w \cdot u' - u \cdot w'}{w^2} \quad ; \quad \text{onde:}$$

$$\begin{cases} u = \operatorname{sen} x \Rightarrow u' = \cos x \cdot x' \\ w = \cos x \Rightarrow w' = -\operatorname{sen} x \cdot x' \end{cases}$$

Assim:

$$y' = \frac{w \cdot u' - u \cdot w'}{w^2} = \frac{\cos x \cdot \cos x \cdot x' - \operatorname{sen} x \cdot (-\operatorname{sen} x \cdot x')}{(\cos x)^2}$$

$$= \frac{\cos^2 x \cdot x' + \operatorname{sen}^2 x \cdot x'}{\cos^2 x} =$$

$$= \frac{(\overbrace{\cos^2 x + \operatorname{sen}^2 x}^{=1}) \cdot x'}{\cos^2 x} = \frac{1}{\cos^2 x} \cdot x'$$

pois da Trigonometria:
$\operatorname{sen}^2 x + \cos^2 x = 1$

$$= \sec^2 x \cdot x'$$

pois $\sec x = \frac{1}{\cos x}$.

Ex: $y = \tan(x^4 - x)$. $\quad y' = ?$

$$y = \tan u \Rightarrow y' = \sec^2 u \cdot u'$$

$$u = x^4 - x \Rightarrow u' = 4x^3 - 1$$

Assim:

$$y' = \sec^2(x^4 - x) \cdot (4x^3 - 1)$$

$$\boxed{y' = (4x^3 - 1) \cdot \sec^2(x^4 - x)}$$

EXERCÍCIOS DIVERSOS:

Calcule a derivada de cada função:

(a) $y = x^3 + \cos(x - 1)$

(b) $y = \sqrt{x^2 + 2x - 3}$

(c) $y = \ln(\operatorname{sen}(x^2 + 1))$

Solução: (a) $y = x^3 + \cos(x-1) = u + v$

$y' = u' + v'$; onde:

$$\begin{cases} u = x^3 \Rightarrow u' = 3x^2 \\ v = \cos(x-1) \Rightarrow v' = -\text{sen}(x-1) \cdot 1 \end{cases}$$

$(\cos w)' = -\text{sen}\, w \cdot w'$

$\Rightarrow y' = 3x^2 - \text{sen}(x-x)$.

(b) $y = \sqrt{x^2+2x-3} = (x^2+2x-3)^{\frac{1}{2}} = v^k$

$k = \frac{1}{2}$

$y = v^k \Rightarrow y' = k \cdot v^{k-1} \cdot v'$

$v = x^2+2x-3 \Rightarrow v' = 2x+2$

$\Rightarrow y' = k \cdot v^{k-1} \cdot v' = \frac{1}{2} \cdot (x^2+2x-3)^{\frac{1}{2}-1} \cdot (2x+2)$

$= \frac{1}{2} \cdot (2x+2) \cdot (x^2+2x-3)^{-\frac{1}{2}} =$

$$= \frac{2x+2}{2\cdot\sqrt{x^2+2x-3}} = \frac{\cancel{2}(x+1)}{\cancel{2}\cdot\sqrt{x^2+2x-3}}$$

$$y' = \frac{x+1}{\sqrt{x^2+2x-3}}.$$

(c) $y = \ln(\operatorname{sen}(x^2+1))$

$y = \ln u$

$\Rightarrow y' = \dfrac{u'}{u}$

$u = \operatorname{sen}(x^2+1) \Rightarrow u' = \cos(x^2+1).(2x+0)$

$(\operatorname{sen} w)' = \cos w . w'$

$\Rightarrow u' = 2x . \cos(x^2+1)$

Assim, vamos obter:

$$y' = \frac{u'}{u} = \frac{2x.\cos(x^2+1)}{\operatorname{sen}(x^2+1)} =$$

$$= 2x . \frac{\cos(x^2+1)}{\operatorname{sen}(x^2+1)} = 2x.\cot(x^2+1)$$

RESUMINDO, VIMOS ATÉ O MOMENTO AS SEGUINTES REGRAS DE DERIVAÇÃO:

01) $y = K \Rightarrow y' = 0$

02) $y = K.x \Rightarrow y' = K$

03) $y = K.v \Rightarrow y' = K.v'$

04) $y = u + v \Rightarrow y' = u' + v'$

05) $y = \dfrac{u}{v} \Rightarrow y' = \dfrac{v.u' - u.v'}{v^2}$

06) $y = \ln v \Rightarrow y' = \dfrac{v'}{v}$

07) $y = e^{v} \Rightarrow y' = e^{v}.v'$

08) $y = \operatorname{sen} v \Rightarrow y' = \cos v \cdot v'$

09) $y = \cos v \Rightarrow y' = -\operatorname{sen} v \cdot v'$

10) $y = \tan v \Rightarrow y' = \sec^2 v \cdot v'$