ÁLGEBRA LINEAR I

28/10/25 - AULA 11

Na aula passada definimos a matriz adjunta de uma matriz quadrada, $A_{n\times n}$, dada por:

$$\text{adj}\,A = \begin{pmatrix} A_{11} & A_{21} & \vdots & A_{n1} \\ A_{12} & A_{22} & \vdots & A_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ A_{1n} & A_{2n} & \vdots & A_{nn} \end{pmatrix};$$

onde

$$A_{ij} = (-1)^{i+j} \cdot \det(M_{ij})$$

Vimos também que se A for inversível, a sua inversa A^{-1} pode ser determinada calculando:

$$A^{-1} = \frac{1}{\det A} \cdot \text{adj}\,A.$$

Ex.: $A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 2 & 2 \\ -1 & 2 & 3 \end{pmatrix}.$

Verifique se A é inversível. Se for, obtenha sua inversa a partir da matriz adjunta.

Solução:

$$\det(A) = \begin{vmatrix} 1 & 2 & -1 \\ 0 & 2 & 2 \\ -1 & 2 & 3 \end{vmatrix} \begin{matrix} 1 & 2 \\ 0 & 2 \\ -1 & 2 \end{matrix}$$

$$= 6 - 4 + 0 - 2 - 4 - 0 = -4 \neq 0.$$

Logo, A é invensível.

Então, $\exists\, A^{-1} = \dfrac{1}{\det A} \cdot \operatorname{adj} A$, onde:

$$\operatorname{adj} A = \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix},$$ sendo:

- $A_{11} = (-1)^{1+1} \cdot \det(M_{11}) = + \begin{vmatrix} 2 & 2 \\ 2 & 3 \end{vmatrix} = 6 - 4 = 2$

- $A_{12} = (-1)^{1+2} \cdot \det(M_{12}) = - \begin{vmatrix} 0 & 2 \\ -1 & 3 \end{vmatrix} = 0 - (+2) = -2$

- $A_{13} = (-1)^{1+3} \cdot \det(M_{13}) = +\begin{vmatrix} 0 & 2 \\ -1 & 2 \end{vmatrix} = 0 + 2 = \underline{\underline{2}}$

- $A_{21} = \underbrace{(-1)^{2+1}}_{-} \cdot \det(M_{21}) = -\begin{vmatrix} 2 & -1 \\ 2 & 3 \end{vmatrix} = -(6+2) = \underline{\underline{-8}}$

- $A_{22} = \underbrace{(-1)^{2+2}}_{+} \cdot \det(M_{22}) = \begin{vmatrix} 1 & -1 \\ -1 & 3 \end{vmatrix} = 3 - 1 = \underline{\underline{2}}$

- $A_{23} = \underbrace{(-1)^{2+3}}_{-} \cdot \det(M_{23}) = -\begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix} = -(2+2) = \underline{\underline{-4}}$

- $A_{31} = (-1)^{3+1} \cdot \det(M_{31}) = +\begin{vmatrix} 2 & -1 \\ 2 & 2 \end{vmatrix} = 4 + 2 = \underline{\underline{6}}$

- $A_{32} = (-1)^{3+2} \cdot \det(M_{32}) = -\begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} = -2$

- $A_{33} = (-1)^{3+3} \cdot \det(M_{33}) = +\begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} = \underline{\underline{2}}$

Assim, obtemos:

$$A^{-1} = \frac{1}{\det A} \cdot \operatorname{adj} A$$

$$= \frac{1}{-4} \cdot \begin{pmatrix} 2 & -8 & 6 \\ -2 & 2 & -2 \\ 2 & -4 & 2 \end{pmatrix} =$$

$$= \begin{pmatrix} -\frac{2}{4} & +\frac{8}{4} & -\frac{6}{4} \\ \frac{2}{4} & -\frac{2}{4} & \frac{2}{4} \\ -\frac{2}{4} & \frac{4}{4} & -\frac{2}{4} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{2} & 2 & -\frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \end{pmatrix}$$

TEOREMA (REGRA DE CRAMER) Seja um sistema linear da forma $Ax = b$, onde

$$A = (a_{ij})_{n\times n} \; ; \; b = (b_1 \; b_2 \; \dots \; b_n)^t \text{ e}$$

$x = (x_1 \; x_2 \; \dots \; x_n)^t$. Se o sistema possui solução única, ela é dada por:

$$x_i = \frac{\det A_i}{\det A},$$

onde $\det A_i$ é o determinante da matriz obtida de A, substituindo a coluna i pela matriz coluna $b = (b_1 \; b_2 \; \dots \; b_n)^t$.

DEMONSTR.: Suponha o sistema $Ax = b$ nas hipóteses do teorema. Logo, A é inversível. Assim, $\exists\, A^{-1}$ inversa de A.

obs: lembre que o sistema $Ax = b$, se

expandido, fica:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} a_{11} \cdot x_1 + a_{12} x_2 + \cdots + a_{1n} x_n = b_1 \\ a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n = b_2 \\ \cdots\cdots \\ a_{n1} x_1 + a_{n2} x_2 + \cdots + a_{nn} x_n = b_n \end{cases}$$

Multiplicando à esquerda por A^{-1}, vem:

$$A x = b$$

$$A^{-1} \cdot (A x) = A^{-1} \cdot b$$

$$\Rightarrow \underbrace{(A^{-1} \cdot A)}_{= I} \cdot x = A^{-1} \cdot b$$

$$x = \underbrace{A^{-1}}_{= \frac{1}{\det A} \cdot \operatorname{adj} A} \cdot b$$

$$\Rightarrow \quad \underset{\sim}{x} = \frac{1}{\det A} \cdot \operatorname{adj} A \cdot b = \frac{\operatorname{adj} A \cdot b}{\det A} \; ;$$

onde:

$$\operatorname{adj} A \cdot b = \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$= \begin{pmatrix} A_{11} \cdot b_1 + A_{21} \cdot b_2 + \cdots + A_{n1} \cdot b_n \\ A_{12} \cdot b_1 + A_{22} \cdot b_2 + \cdots + A_{n2} \cdot b_n \\ \cdots \\ A_{1n} \cdot b_1 + A_{2n} b_2 + \cdots A_{nn} \cdot b_n \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{k=1}^{n} A_{k1} \cdot b_k \\ \sum_{k=1}^{n} A_{k2} \cdot b_k \\ \vdots \\ \sum_{k=1}^{n} A_{kn} \cdot b_k \end{pmatrix}$$

Defina a matriz A_i por:

$$A_i = \begin{pmatrix} a_{11} & a_{12} \dots & a_{1,i-1} & b_1 & a_{1,i+1} \dots & a_{1n} \\ a_{21} & a_{22} \dots & a_{2,i-1} & b_2 & a_{2,i+1} \dots & a_{2n} \\ - & - & - & - & - & - \\ a_{n1} & a_{n2} \dots & a_{n,i-1} & b_n & a_{n,i+1} \dots & a_{nn} \end{pmatrix}$$

i-ésima coluna de A é substituída pela matriz coluna $b = (b_1\ b_2 \dots b_n)^t$

Desenvolvendo o $\det A_i$ sobre esta i-ésima coluna, vamos obter:

$$\det A_i = b_1 . A_{1i} + b_2 . A_{2i} + \dots + b_n . A_{ni} = \sum_{k=1}^{n} b_k . A_{ki}$$

Assim; tem-se que:

$$x = \frac{\operatorname{adj} A . b}{\det A} = \frac{1}{\det A} . \begin{pmatrix} \sum_{k=1}^{n} A_{k1} b_k \\ \vdots \\ \sum_{k=1}^{n} A_{kn} . b_k \end{pmatrix}$$

$$\Rightarrow \quad x_i = \frac{\overbrace{\sum_{k=1}^{n} b_k \cdot A_{ki}}^{=\det A_i}}{\det A} = \frac{\det A_i}{\det A}$$

$\square$

Ex.: Resolver, usando a regra de Cramer, o sistema linear:

$$\begin{cases} 2x + y - z = -6 \\ x + 3y + z = 4 \\ x + y - z = 4 \end{cases}$$

Solução:

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 3 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

$$\Rightarrow \det A = \left| \begin{array}{ccc|cc} 2 & 1 & -1 & 2 & 1 \\ 1 & 3 & 1 & 1 & 3 \\ 1 & 1 & -1 & 1 & 1 \end{array} \right| =$$

$$= -6 \cancel{+1} \cancel{-1} + 3 - 2 + 1 = -4$$

$$\det A_1 = \left|\begin{matrix} -6 & 1 & -1 \\ 4 & 3 & 1 \\ 4 & 1 & -1 \end{matrix}\right.\left|\begin{matrix} -6 & 1 \\ 4 & 3 \\ 4 & 1 \end{matrix}\right.$$

$$= 18 + \cancel{4} - \cancel{4} + 12 + 6 + 4 = \underline{\underline{40}}$$

$$\det A_2 = \left|\begin{matrix} 2 & -6 & -1 \\ 1 & 4 & 1 \\ 1 & 4 & -1 \end{matrix}\right.\left|\begin{matrix} 2 & -6 \\ 1 & 4 \\ 1 & 4 \end{matrix}\right. =$$

$$= -8 - 6 - \cancel{4} + \cancel{4} - 8 - 6 = \underline{\underline{-28}}$$

$$\det A_3 = \left|\begin{matrix} 2 & 1 & -6 \\ 1 & 3 & 4 \\ 1 & 1 & 4 \end{matrix}\right.\left|\begin{matrix} 2 & 1 \\ 1 & 3 \\ 1 & 1 \end{matrix}\right.$$

$$= 24 + 4 - 6 + 18 - 8 - 4 = \underline{\underline{28}}$$

Portanto, obtemos:

$$\underset{\sim}{x} = \frac{\det A_1}{\det A} = \frac{40}{-4} = \underset{\sim}{-10}$$

$$y = \frac{\det A_2}{\det A} = \frac{-28}{-4} = 7$$

$$z = \frac{\det A_3}{\det A} = \frac{28}{-4} = -7$$

$$S = \{(x, y, z)\} = \{(-10, 7, -7)\}.$$