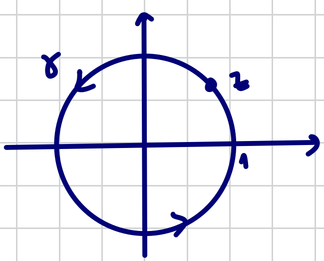
VARIÁVEIS COMPLEXAS $\boxed{15/08/25 \text{ - AULA 30}}$

APLICAÇÃO DO TEOREMA DOS RESÍDUOS: INTEGRAIS REAIS.

Como aplicação do teorema dos resíduos, há dois tipos de integrais definidas: as racionais em termos de senos e cossenos e as impróprias

1º: integrais da forma $\int_0^{2\pi} R(\operatorname{sen}\theta, \cos\theta)\, d\theta$, onde $R(\operatorname{sen}\theta, \cos\theta)$ denota uma função racional em termos de seno e cosseno.

Neste caso, considere $\gamma = \partial D_1(0)$.



$$\int_0^{2\pi} R(\operatorname{sen}\theta, \cos\theta)\, d\theta$$

Escreva $z = e^{i\theta}$

- $\operatorname{sen}\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i} = \dfrac{z - z^{-1}}{2i} =$

$$= \frac{z - \frac{1}{z}}{2i} = \underbrace{\frac{1}{2i}\left(z - \frac{1}{z}\right)}$$

$$\bullet\ \underbrace{\cos\theta} = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + z^{-1}}{2}$$

$$= \underbrace{\frac{1}{2}\left(z + \frac{1}{z}\right)}$$

Além disso, como $z = e^{i\theta}$, então

$$dz = i \cdot e^{i\theta} \cdot d\theta$$

$$\Rightarrow \underbrace{d\theta} = \frac{dz}{i \cdot \underbrace{e^{i\theta}}_{z}} \Rightarrow \underbrace{d\theta = \frac{dz}{i \cdot z}}$$

Assim, escrevemos:

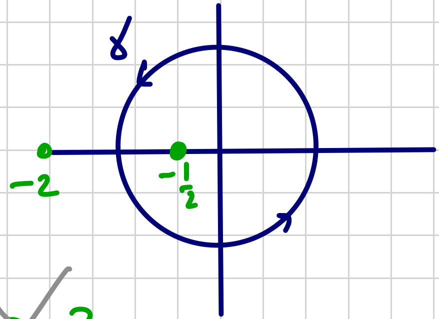
$$\int_0^{2\pi} R(\operatorname{sen}\theta, \cos\theta)\, d\theta = \int_{\partial D_1(0)} R(z)\, dz$$

Ex.:

$$\int_0^{2\pi} \frac{d\theta}{10 + 8\cos\theta} = \int_{\partial D_1(0)} \frac{\frac{dz}{i \cdot z}}{10 + \cancel{8}^{4} \cdot \frac{1}{\cancel{2}}\left(z + \frac{1}{z}\right)}$$

$$= \int_{\partial D_1(0)} \frac{\frac{dz}{i\cancel{z}}}{\frac{10z+4z^2+4}{\cancel{z}}} = \frac{1}{4i}\int_{\partial D_1(0)} \frac{dz}{z^2+\frac{5}{2}z+1} =$$

$$= \frac{1}{4i}\int_{\partial D_1(0)} \frac{dz}{(z+\frac{1}{2})\cdot(z+2)} =$$



SINGULARIDADES:

$z=-\frac{1}{2}$ $\quad \cancel{z=-2}$ está fora do interior de γ.

PÓLO SIMPLES.

$$f(z) = \frac{1}{(z+\frac{1}{2})(z+2)} = \frac{A}{z+\frac{1}{2}} + \frac{B}{z+2}$$

$$\frac{A}{z+\frac{1}{2}} + \sum_{n=0}^{\infty} a_n\left(z+\frac{1}{2}\right)^n$$

Então, pelo T. dos resíduos:

$$= \frac{1}{4i}\left(2\pi i \operatorname*{res}_{z=-\frac{1}{2}} f(z)\right) \text{; onde}$$

$$\operatorname*{res}_{z=-\frac{1}{2}} f(z) = A = \lim_{z\to -\frac{1}{2}} \left(z+\frac{1}{2}\right)\cdot f(z) =$$

$$= \lim_{z \to -\frac{1}{2}} \cancel{\left(z+\frac{1}{2}\right)} \cdot \frac{1}{\cancel{\left(z+\frac{1}{2}\right)}(z+2)} =$$

$$= \frac{1}{-\frac{1}{2}+2} = \frac{2}{3}$$

Portanto;

$$\int_0^{2\pi} \frac{d\theta}{10+8\cos\theta} = \frac{1}{4i}\int_\gamma \frac{dz}{\left(z+\frac{1}{2}\right)(z+2)} = \frac{1}{4\cancel{i}} \cdot \Big(2\pi \cancel{i}\, \overbrace{\operatorname{Res}_{z=-\frac{1}{2}} f(z)}^{2/3}\Big)$$

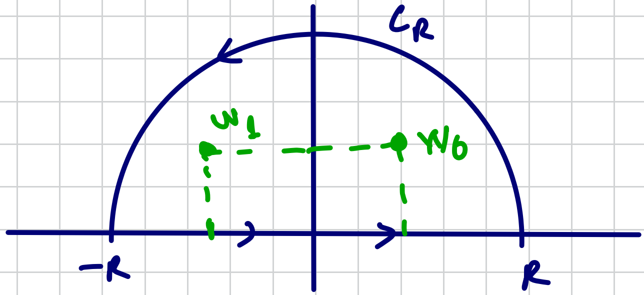
$$= \frac{\pi}{\cancel{2}} \cdot \frac{\cancel{2}}{3} = \frac{\pi}{3}$$

2º: integrais impróprios: neste caso considere-se uma função complexa a partir da função real dada e considera-se $\gamma = C_R \cup [-R,R]$; onde C_R é a semi-circunferência nos 1º e 2º quadrantes e $R > 0$.

Ex.: (a) $\displaystyle\int_{-\infty}^{+\infty} \frac{x^2\,dx}{x^4+1} = ?$

Considere $f(z) = \dfrac{z^2}{z^4+1}$ e seja

$\gamma = C_R \cup [-R, R]$:



singularidades: são os zeros de $z^4+1=0$

$\Updownarrow$

$z = \sqrt[4]{-1}$

Aplicando a fórmula da raiz (quarta) de De Moivre, obtemos os zeros de $z^4+1=0$:

$\underline{\underline{w_0}} = \dfrac{1+i}{\sqrt{2}}$; $\underline{\underline{w_1}} = \dfrac{-1+i}{\sqrt{2}}$;

$w_2 = \dfrac{-1-i}{\sqrt{2}}$; $w_3 = \dfrac{1-i}{\sqrt{2}}$.

Então,

$$f(z) = \frac{z^2}{z^4+1} = \frac{z^2}{(z-w_0)(z-w_1)(z-w_2)(z-w_3)}$$

Note que apenas w_0 e w_1 pertencem ao interior da curva γ. Então, apenas estes entrarão no cálculo do resíduo.

Ambos são pólos simples.

Então:

- $$\operatorname*{res}_{z=w_0} f(z) = \lim_{z\to w_0} (z-w_0)\cdot f(z)$$

$$= \lim_{z\to w_0} \cancel{(z-w_0)}\cdot \frac{z^2}{\cancel{(z-w_0)}(z-w_1)(z-w_2)(z-w_3)}$$

$$= \lim_{z\to \frac{1+i}{\sqrt{2}}} \frac{z^2}{(z-w_1)(z-w_2)(z-w_3)} =$$

$$= \frac{\left(\frac{1+i}{\sqrt{2}}\right)^2}{\left(\frac{1+i}{\sqrt{2}} - \left[-\frac{1+i}{\sqrt{2}}\right]\right)\cdot\left(\frac{1+i}{\sqrt{2}} - \left[-\frac{1-i}{\sqrt{2}}\right]\right)\cdot\left(\frac{1+i}{\sqrt{2}} - \left[\frac{1-i}{\sqrt{2}}\right]\right)}$$

$$= \frac{\frac{\cancel{1} + 2i \cancel{-1}}{2}}{\left(\frac{1+\cancel{i}+1-\cancel{i}}{\sqrt{2}}\right)\cdot\left(\frac{1+i+1+i}{\sqrt{2}}\right)\cdot\left(\frac{\cancel{1}+i-\cancel{1}+i}{\sqrt{2}}\right)}$$

$$= \frac{\cancel{i}}{\frac{2}{\cancel{\sqrt{2}}}\cdot\frac{2(1+i)}{\cancel{\sqrt{2}}}\cdot\frac{2\cancel{i}}{\sqrt{2}}} = \frac{1}{\frac{4}{\sqrt{2}}(1+i)}$$

$$= \frac{\sqrt{2}}{4(1+i)} = \frac{\sqrt{2}(1-i)}{8} = \frac{\sqrt{2}-\sqrt{2}i}{8}$$

• $\operatorname{res}_{z=w_1} f(z) = \lim_{z\to w_1} (z-w_1)\, f(z) =$

$$= \lim_{z\to w_1} \cancel{(z-w_1)} \cdot \frac{z^2}{(z-w_0)\,\cancel{(z-w_1)}\cdot(z-w_2)\cdot(z-w_3)}$$

$$= \frac{\left(\frac{-1+i}{\sqrt{2}}\right)^2}{\left(\frac{-1+i}{\sqrt{2}} - \left[\frac{1+i}{\sqrt{2}}\right]\right)\cdot\left(\frac{-1+i}{\sqrt{2}} - \left[\frac{-1-i}{\sqrt{2}}\right]\right)\cdot\left(\frac{-1+i}{\sqrt{2}} - \left[\frac{1-i}{\sqrt{2}}\right]\right)}$$

$$= \frac{\frac{\cancel{1} - 2i - \cancel{1}}{2}}{\left(\frac{-1 \cancel{+i} - 1 \cancel{-i}}{\sqrt{2}}\right)\cdot\left(\frac{\cancel{-1} + i \cancel{+1} + i}{\sqrt{2}}\right)\cdot\left(\frac{-1+i-1+i}{\sqrt{2}}\right)}$$

$$= \frac{-\cancel{i}}{-\frac{\cancel{2}}{\cancel{\sqrt{2}}}\cdot\frac{2\cancel{i}}{\cancel{\sqrt{2}}}\cdot\frac{2(-1+i)}{\sqrt{2}}} = \frac{-1}{-\frac{4(-1+i)}{\sqrt{2}}} =$$

$$= +\frac{\sqrt{2}}{4(-1+i)} = \frac{-\sqrt{2} - \sqrt{2}\, i}{8} \quad //$$

Pelo T. dos resíduos:

$$\int_\gamma f(z)\,dz = 2\pi i \cdot \left(\operatorname*{Res}_{z=w_0} f(z) + \operatorname*{Res}_{z=w_1} f(z) \right)$$

$$= 2\pi i \cdot \left(\cancel{\frac{\sqrt{2}}{8}} - \frac{\sqrt{2}}{8} i + \cancel{\frac{-\sqrt{2}}{8}} - \frac{\sqrt{2}}{8} i \right)$$

$$= \cancel{2}\pi i \cdot \left(-\frac{\sqrt{2}}{\cancel{4}} i \right) = + \frac{\sqrt{2}\pi}{2}$$

$$\Rightarrow \underbrace{\int_{C_R} f(z)\,dz}_{\text{VAMOS MOSTRAR QUE ESTA INTEGRAL VAI A ZERO QUANDO } R \to \infty.} + \int_{-R}^{R} f(z)\,dz = \int_\gamma f(z)\,dz = \frac{\sqrt{2}}{2}\pi$$

VAMOS MOSTRAR QUE ESTA INTEGRAL VAI A ZERO QUANDO $R \to \infty$.

$$\left| \int_{C_R} f(z)\,dz \right| \leq \int_{C_R} |f(z)| \cdot |dz| = \int_{C_R} \frac{|z^2|}{|z^4+1|} \cdot |dz| =$$

$$= \int_{C_R} \frac{|z|^2}{|z^4+1|} \cdot |dz| \leq \int_{C_R} \frac{R^2}{R^4-1} \cdot |dz| =$$

$z \in C_R \Rightarrow |z| = R.$

$$|z^4+1| \geq |z^4| - 1 = |z|^4 - 1 = R^4 - 1$$

$$= \frac{R^2}{R^4-1} \int_{C_R} |dz| = \frac{R^2}{R^4-1} \cdot \pi R =$$

lembrai-vos: C_R e' meia circunferência

$$= \frac{\pi R^3}{R^4-1} \xrightarrow[R \to +\infty]{} 0$$

Assim, passando limite com $R \to \infty$, vamos obter:

$$\underbrace{\lim_{R\to\infty}\int_{C_R} f(z)\,dz}_{0} + \underbrace{\lim_{R\to\infty}\int_{-R}^{R} f(z)\,dz}_{\int_{-\infty}^{+\infty} f(z)\,dz;} = \underbrace{\lim_{R\to\infty}\frac{\sqrt{2}}{2}\pi}_{=\frac{\sqrt{2}}{2}\pi}$$

onde $z = x + 0i = x$

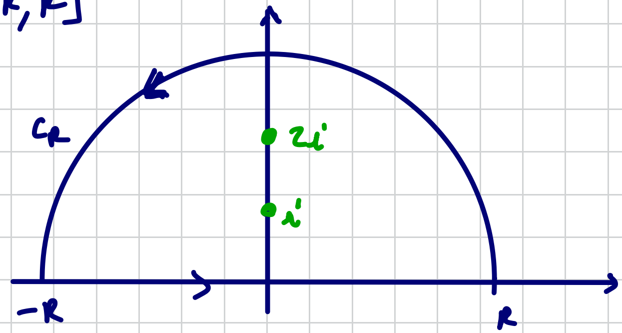
conclusão: $\displaystyle\int_{-\infty}^{+\infty} f(x)\,dx = \frac{\sqrt{2}}{2}\pi$

(b) $\displaystyle\int_{-\infty}^{+\infty} \frac{dx}{(x^2+1)^2(x^2+4)} = ?$

Considere $f(z) = \dfrac{1}{(z^2+1)^2(z^2+4)}$.

temos:
- $z = i$ e $z = -i$ – pólos de ordem 2
- $z = 2i$ e $z = -2i$ – pólos simples.

$$\gamma = C_R \cup [-R, R]$$



$$\int_\gamma f(z)\,dz = 2\pi i \cdot \left(\operatorname*{Res}_{z=i} f(z) + \operatorname*{Res}_{z=2i} f(z) \right)$$

- $$\operatorname*{Res}_{z=2i} f(z) = \lim_{z \to 2i} (z-2i) \cdot f(z) =$$

2i é polo simples

$$= \lim_{z \to 2i} (z-\cancel{2i}) \cdot \frac{1}{(z^2+1)^2 (z+2i)(z-\cancel{2i})}$$

$$= \frac{1}{(\underbrace{(2i)^2}_{-4}+1)^2 \cdot (2i+2i)} = \frac{1}{9 \cdot 4i}$$

$$= \frac{1}{36i}$$

$$\bullet \ \underset{z=i}{\operatorname{Res}} f(z) = \frac{1}{1!} \lim_{z \to i} \frac{d}{dz}\left[\cancel{(z-i)^2} \cdot \frac{1}{\cancel{(z-i)^2}(z+i)^2 \cdot (z^2+4)}\right]$$

pólo de ordem 2

i

pólo de ordem m:

$$\underset{z \to z_0}{\operatorname{Res}} f(z) = \frac{1}{(m-1)!} \cdot \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m \cdot f(z)\right]$$

$$= \lim_{z \to i} \frac{d}{dz}\left(\frac{1}{(z+i)^2(z^2+4)}\right) = \lim_{z \to i} \frac{d}{dz}\left((z+i)^{-2} \cdot (z^2+4)^{-1}\right)$$

$$= \lim_{z \to i} \left[(z+i)^{-2} \cdot (-1) \cdot (z^2+4)^{-2} \cdot 2z + (-2) \cdot (z+i)^{-3} \cdot 1 \cdot (z^2+4)^{-1}\right]$$

$$= \lim_{z \to i} \left[\frac{-2z}{(z+i)^2(z^2+4)^2} - \frac{2}{(z+i)^3 \cdot (z^2+4)}\right]$$

$$= \lim_{z \to i} \frac{-2}{(z+i)^2 \cdot (z^2+4)} \cdot \left[\frac{z}{z^2+4} + \frac{1}{z+i}\right]$$

$$= \frac{-\cancel{2}}{-4 \cdot 3} \cdot \left[\frac{i}{3} + \frac{1}{2i}\right]$$

$$= \frac{1}{6} \cdot \left(\frac{-2+3}{6i} \right) = \frac{1}{36i}$$

$$\Rightarrow \int_\gamma f(z)\,dz = 2\pi i \cdot \left(\operatorname*{Res}_{z=2i} f(z) + \operatorname*{Res}_{z=i} f(z) \right)$$

$$= 2\pi i \left(\frac{1}{36i} + \frac{1}{36i} \right)$$

$$= 2\pi \cancel{i} \cdot \left(\frac{2}{36\cancel{i}} \right) = \frac{\pi}{9}$$

$$\Rightarrow \underbrace{\int_{C_R} f(z)dz}_{} + \int_{-R}^{R} f(z)\,dz = \int_\gamma f(z)\,dz = \frac{\pi}{9} \qquad (*)$$

MOSTRAR QUE VAI A ZERO QUANDO $R \to \infty$:

$$\left| \int_{C_R} f(z)dz \right| \leq \int_{C_R} |f(z)| \cdot |dz| = \int_{C_R} \frac{1}{|z^2+1|^2 \cdot |z^2+4|} \cdot |dz|$$

Como $|z^2+1| \geq |z|^2 - 1 \underset{\uparrow}{=} R^2 - 1$; e

$\boxed{|z|=R}$

$$|z^2+4| \geq |z|^2 - 4 = R^2-4 \,;$$

então:

$$\left|\int_{C_R} f(z)\,dz\right| \leq \int_{C_R} \frac{|dz|}{(R^2-1)^2\cdot(R^2-4)} =$$

$$= \frac{1}{(R^2-1)^2(R^2-4)} \cdot \int_{C_R} |dz| = \frac{\pi R}{(R^2-1)^2(R^2-4)} \xrightarrow[R\to\infty]{} 0$$

Então, passando (*) ao limite com $R \to \infty$, obtemos:

$$0 + \int_{-\infty}^{+\infty} f(x)\,dx = \frac{\pi}{9}$$

Fim da Disciplina.