CÁLCULO II - AULA DE EXERCÍCIOS 20/08/25

LS1 02-(j):

(j) $$\int \frac{(4x^2+6x)dx}{(x-1)^2(x+1)} = -\frac{5}{x-1} + \frac{9}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + c$$

$$\frac{4x^2+6x}{(x-1)^2(x+1)} = \frac{A}{(x-1)^2} + \frac{B}{x-1} + \frac{C}{x+1}$$

$$= \frac{A(x+1) + B(x+1)(x-1) + C(x-1)^2}{(x-1)^2(x+1)}$$

$$4x^2+6x = Ax + A + Bx^2 - B + Cx^2 - 2Cx + C$$

$$\begin{cases} B + C = 4 \\ A - 2C = 6 \\ A - B + C = 0 \end{cases}$$

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$$2A = 10 \rightarrow \boxed{A = 5}$$

$$\begin{cases} A - 2C = 6 \\ -2C = 6 - A \\ -2C = 6 - 5 \end{cases}$$

$$\boxed{C = -\frac{1}{2}}$$

$$B + C = 4$$

$$B = 4 - C = 4 - \left(-\frac{1}{2}\right)$$

$$B = 4 + \frac{1}{2} \Longrightarrow \boxed{B = \frac{9}{2}}$$

Disso, vamos obter:

$$\int \frac{4x^2+6x}{(x-1)^2\cdot(x+1)}\,dx = \int\left(\frac{5}{(x-1)^2} + \frac{\frac{9}{2}}{x-1} + \frac{-\frac{1}{2}}{x+1}\right)dx$$

$$= 5\cdot\int (x-1)^{-2}\,dx + \frac{9}{2}\int \frac{dx}{x-1} - \frac{1}{2}\int\frac{dx}{x+1}$$

$\int u^k\,du$

$u = x-1$

$\hookrightarrow du = dx$

$\int \frac{du}{u} = \ln|u| + c$

$$= 5\cdot\frac{(x-1)^{-1}}{-1} + \frac{9}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + c$$

$$= -\frac{5}{x-1} + \frac{9}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C.$$

L5)

3. Mostre que:

(a) $\displaystyle\int_2^4 \frac{x^3-2}{x^3-x^2}dx = \frac{5}{2} + \ln\frac{4}{3}$

como o grau do numerador é maior ou igual ao do denominador, precisamos efetuar uma divisão, para a fração ficar própria.

$$\begin{array}{r|l} \cancel{x^3} - 2 & x^3 - x^2 \\ \cline{2-2} -\cancel{x^3} + x^2 & 1 \\ \hline x^2 - 2 & \end{array}$$

Assim:

$$x^3-2 = 1.(x^3-x^2) + x^2-2$$

Disso:

$$\frac{x^3-2}{x^3-x^2} = \frac{1.(x^3-x^2)+x^2-2}{x^3-x^2}$$

$$= \frac{\cancel{x^3-x^2}}{\cancel{x^3-x^2}} + \frac{x^2-2}{x^3-x^2} =$$

$$= 1 + \frac{x^2-2}{x^3-x^2}$$

Assim, teremos: (vamos, primeiramente, achar a integral indefinida. Depois, aplicamos o T.F.C.)

$$\int \frac{x^3-2}{x^3-x^2}\,dx = \int \left(1 + \frac{x^2-2}{x^3-x^2}\right)dx =$$

$$= \int dx + \int \frac{x^2-2}{x^3-x^2}\,dx =$$

$$= x + \int \frac{x^2-2}{x^2(x-1)}\,dx$$

$$\frac{x^2-2}{x^2(x-1)} = \frac{A}{x^2} + \frac{B}{x} + \frac{C}{x-1}$$

$$= \frac{A\cdot(x-1) + Bx(x-1) + Cx^2}{x^2(x-1)}$$

$$\Leftrightarrow 1x^2 - 2 = Ax - A + Bx^2 - Bx + Cx^2$$

$$\begin{cases} B + C = 1 \\ A - B = 0 \\ -A = -2 \end{cases} \Rightarrow \boxed{A = 2}$$

$$2 - B = 0 \Rightarrow \boxed{B = 2}$$

$$\begin{cases} B + C = 1 \\ C = 1 - B \\ C = 1 - 2 \\ \boxed{C = -1} \end{cases}$$

Assim, obtemos:

$$\int f(x)\,dx = x + \int \frac{(x-2)\,dx}{x^2(x-1)} =$$

$$= x + \int \left(\frac{2}{x^2} + \frac{2}{x} + \frac{-1}{x-1} \right) dx =$$

$$= x + \underbrace{2\int x^{-2}\,dx}_{\int u^{n}\,du} + \underbrace{2\int \frac{dx}{x} - \int \frac{dx}{x-1}}_{\int \frac{du}{u}} .$$

$$= x + 2 \cdot \frac{x^{-1}}{-1} + 2 \cdot \ln|x| - \ln|x-1| + C$$

$$= x - \frac{2}{x} + 2\ln|x| - \ln|x-1| + C.$$

Por fim, obtemos:

$$\int_2^4 \frac{x^3-2}{x^3-x^2}\,dx = \left(x - \frac{2}{x} + 2\ln|x| - \ln|x-1| \right) \Bigg|_2^4 =$$

$$= 4 - \frac{2}{4} + 2\ln 4 - \ln 3 - \left(2 - \cancel{\frac{2}{2}} + 2\ln 2 - \underbrace{\ln 1}_{=0} \right)$$

$$= 4 - \frac{1}{2} + 2\ln 4 - \ln 3 - 2 + 1 - 2\ln 2 + 0$$

$$= 3 - \frac{1}{2} + 2\ln 2^2 - \ln 3 - 2\ln 2$$

$$= \frac{5}{2} + 4\ln 2 - \ln 3 - 2\ln 2$$

$$= \frac{5}{2} + 2\ln 2 - \ln 3 = \frac{5}{2} + \ln 2^2 - \ln 3 = \frac{5}{2} + \ln\frac{4}{3}$$

De uma prova antiga, de 2018:

Questão 01. Calcule cada integral indefinida a seguir.

(a) $\displaystyle\int \frac{2x\,dx}{(x^2+1)(x+1)^2}$ (b) $\displaystyle\int \frac{dx}{3\cos x - 2\operatorname{sen} x + 3}$ (c) $\displaystyle\int \frac{dx}{\sqrt[3]{x^2} + \sqrt[4]{x^3}}$

(b) $\displaystyle\int \frac{dx}{3\cos x - 2\operatorname{sen} x + 3}$

Usaremos a troca universal:

$$\tan\frac{x}{2} = z$$

$$\operatorname{sen} x = \frac{2z}{1+z^2} \qquad dx = \frac{2dz}{1+z^2}$$

$$\cos x = \frac{1-z^2}{1+z^2}$$

$$\Rightarrow \int \frac{dx}{3\cos x - 2\operatorname{sen} x + 3} = \int \frac{\frac{2dz}{1+z^2}}{3 \cdot \frac{(1-z^2)}{1+z^2} - 2 \cdot \frac{2z}{1+z^2} + 3}$$

$$= \int \frac{\frac{2\,dz}{\cancel{1+z^2}}}{\frac{3 - 3\cancel{z^2} - 4z + 3 + 3\cancel{z^2}}{\cancel{1+z^2}}} = \int \frac{2dz}{6-4z} =$$

$$= \int \frac{dz}{3-2z} = \int \frac{-\frac{dr}{2}}{r} = -\frac{1}{2}\int \frac{dr}{r} = -\frac{1}{2}\ln|r| + C \;=$$

$$\int \frac{dr}{r} = \ln|r| + C$$

$$r = \underline{3-2z} \Rightarrow dr = -2dz$$

$$\hookrightarrow dz = -\frac{dr}{2}$$

$$= -\frac{1}{2}\ln|3-2z| + C = -\frac{1}{2}\ln\left|3 - 2\cdot\tan\left(\frac{x}{2}\right)\right| + C$$

$$z = \tan\frac{x}{2}$$

(c) $\displaystyle\int \frac{dx}{\sqrt[3]{x^2}+\sqrt[4]{x^3}} = ?$

$\text{mmc}(3,4) = 12$

Escreve $\underline{x = p^{12}} \Rightarrow dx = 12p^{11}\cdot dp.$

Assim, temos:

$$\int \frac{dx}{\sqrt[3]{x^2}+\sqrt[4]{x^3}} = \int \frac{12p^{11}\cdot dp}{\sqrt[3]{(p^{12})^2}+\sqrt[4]{(p^{12})^3}} =$$

$$\int \frac{12p^{11}\cdot dp}{\sqrt[3]{(p^2)^{\cancel{12}\,4}}+\sqrt[4]{(p^3)^{\cancel{12}}}} = \int \frac{12p^{11}dp}{p^8+p^9} =$$

$$12\int \frac{p^{\cancel{11}\,3}dp}{\cancel{p^8}(1+p)} = 12\int \frac{p^3dp}{p+1}$$

COMO O GRAU DO NUMERADOR É MAIOR OU IGUAL AO DO DENOMINADOR, PRECISAMOS EFETUAR DIVISÃO DE POLINÔMIOS

$$\begin{array}{l|l}
p^3 & p+1 \\ \hline
-\cancel{p^3}-p^2 & p^2-p+1 \\
\quad -\cancel{p^2} & \\
\quad +\cancel{p^2}+p & \\ \hline
\qquad p & \\
\quad -\cancel{p}-1 & \\ \hline
\qquad -1 &
\end{array}$$

$$\Rightarrow p^3 = (p+1)\cdot(p^2-p+1) - 1.$$

Assim:

$$12\int \frac{p^3\,dp}{p+1} = 12\int\left(\frac{(\cancel{p+1})(p^2-p+1)}{\cancel{p+1}} - \frac{1}{p+1}\right)dp$$

$$= 12.\int p^2\,dp - 12\int p\,dp + 12\int dp - 12\int \frac{dp}{p+1}$$

$$= \cancel{12}^{\,4}\frac{p^3}{\cancel{3}} - \cancel{12}^{\,6}\frac{p^2}{\cancel{2}} + 12p - 12.\ln|p+1| + C.$$

$$= 4p^3 - 6p^2 + 12p - 12\ln|p+1| + c =$$

$$x = p^{12} \Rightarrow p = \sqrt[12]{x}$$

$$= 4.\left(\sqrt[\cancel{12}^{\,4}]{x}\right)^{\cancel{3}} - 6.\left(\sqrt[\cancel{12}^{\,6}]{x}\right)^{\cancel{2}} + 12\sqrt[12]{x} - 12\ln\left|\sqrt[12]{x}+1\right| + c$$

$$= 4\sqrt[4]{x} - 6\sqrt[6]{x} + 12\sqrt[12]{x} - 12\ln\left|\sqrt[12]{x}+1\right| + c.$$

De outra prova antiga, de 2018:

Questão 01. Calcule cada integral indefinida a seguir.

(a) $\displaystyle\int \frac{2x\,dx}{(x^2+1)(x+1)^2}$ (b) $\displaystyle\int \frac{dx}{4-5\operatorname{sen} x}$ (c) $\displaystyle\int \frac{\sqrt[6]{x}\,dx}{x-\sqrt[4]{x^3}}$

Questão 02. Calcule a integral imprópria abaixo, se existir.

$$\int_2^{+\infty} \frac{dx}{\sqrt{x-1}}$$

(c) $\displaystyle\int \frac{\sqrt[6]{x}\,dx}{x-\sqrt[4]{x^3}}$. mmc $(6,4) = 12$

Escreva $\underline{x = p^{12}}$. Assim: $dx = 12p^{11}\,dp$

Então:

$$\int \frac{\sqrt[6]{x}\,dx}{x-\sqrt[4]{x^3}} = \int \frac{\overbrace{\sqrt[\cancel{6}]{p^{\cancel{12}}}}^{p^2}\cdot 12p^{11}\,dp}{p^{12}-\sqrt[4]{(p^{12})^3}} =$$

$$= \int \frac{12p^{13}\,dp}{p^{12}-\sqrt[\cancel{4}]{(p^3)^{\cancel{12}\,3}}} = \int \frac{12p^{13}\,dp}{p^{12}-p^9} =$$

$$= \int \frac{12\cancel{p^{13}}\,dp}{\cancel{p^9}(p^3-1)} = 12\int \frac{p^4\,dp}{p^3-1}$$

$$\begin{array}{r|l} p^4 & p^3-1 \\ \hline -p^4+p & p \\ \hline p & \end{array}$$

$$\Rightarrow p^4 = p(p^3-1) + p$$

$$\Rightarrow \frac{p^4}{p^3-1} = \frac{p\cancel{(p^3-1)}}{\cancel{p^3-1}} - \frac{p}{p^3-1}$$

$$= p - \frac{p}{p^3-1}$$

Assim, obtemos:

$$12\int \frac{p^4}{p^3-1}\,dp = 12\int \left(p - \frac{p}{p^3-1}\right)dp =$$

$$= 12\int p\,dp - 12\underbrace{\int \frac{p\,dp}{p^3-1}}_{(*)} = (I)$$

AVALIANDO (*):

$$\int \frac{p}{p^3-1}\,dp = ?$$

$$\begin{cases} p^3-1 = (p-1)(p^2+p+1) \\ a^3-b^3 = (a-b)\cdot(a^2+ab+b^2) \end{cases}$$

$$\int \frac{p\,dp}{p^3-1} = \int \frac{p\,dp}{(p-1)(p^2+p+1)}$$

Agora precisamos decompor em frações parciais.

$$\frac{p}{(p-1)(\underbrace{p^2+p+1}_{\text{Raízes compl.}})} = \frac{A}{p-1} + \frac{Bp+C}{p^2+p+1} =$$

$$= \frac{A\cdot(p^2+p+1) + (Bp+C)\cdot(p-1)}{(p-1)(p^2+p+1)}$$

$$1p = Ap^2 + Ap + A + Bp^2 - Bp + Cp - C$$

$$\begin{cases} A + B = 0 \\ A - B + C = 1 \\ A - C = 0 \end{cases}$$

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$$3A = 1 \Rightarrow A = \frac{1}{3}$$

$$A + B = 0$$

$$B = -A$$

$$B = -\frac{1}{3}$$

$$A - C = 0$$

$$\Rightarrow C = A$$

$$\Rightarrow \boxed{C = \frac{1}{3}}$$

Assim, obtemos:

$$\int \frac{p\,dp}{p^3-1} = \int \frac{p\,dp}{(p-1)(p^2+p+1)} = \int \left(\frac{\frac{1}{3}}{p-1} + \frac{-\frac{1}{3}p + \frac{1}{3}}{p^2+p+1} \right) dp$$

$$= \frac{1}{3}\int \frac{dp}{p-1} + \frac{1}{3}\int \frac{(-p+1)\,dp}{p^2+p+1}$$

$$u = p^2+p+1$$

$$du = (2p+1)\,dp$$

$$= \frac{1}{3}\ln|p-1| + \frac{1}{3}\cdot\int \frac{-\frac{1}{2}(2p+1) + \frac{3}{2}}{p^2+p+1}\,dp$$

$$= \frac{1}{3}\ln|p-1| - \frac{1}{6}\int \frac{(2p+1)\,dp}{p^2+p+1} + \frac{1}{2}\int \frac{dp}{p^2+p+1}$$

$$= \frac{1}{3}\ln|p+1| - \frac{1}{6}\cdot\ln|p^2+p+1| + \frac{1}{2}\int \frac{dp}{p^2+p+1} =$$

completar um quadrado perfeito

$$p^2+p+1 = (p+k)^2+l$$

$$p^2+2kp+k^2+l$$

$$\begin{cases} 2k = 1 \rightarrow k = \frac{1}{2} \\ k^2+l = 1 \end{cases}$$

$$\frac{1}{4}+l = 1 \Rightarrow l = \frac{3}{4}$$

$$\Rightarrow \quad p^2+p+1 = \left(p+\frac{1}{2}\right)^2+\frac{3}{4}$$

$$= \frac{1}{3}\ln|p+1| - \frac{1}{6}\ln|p^2+p+1| + \frac{1}{2}\int \frac{dp}{\left(p+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$\int \frac{du}{u^2+a^2} = \frac{1}{a}\cdot\arctan\left(\frac{u}{a}\right)+c$$

$$= \frac{1}{3}\cdot\ln|p+1| - \frac{1}{6}\ln|p^2+p+1| + \cancel{\frac{1}{2}}\cdot\frac{1}{\frac{\sqrt{3}}{\cancel{2}}}\cdot\arctan\left(\frac{p+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)+c$$

$$= \frac{1}{3}\ln|p+1| - \frac{1}{6}\ln|p^2+p+1| + \frac{1}{\sqrt{3}}\cdot\arctan\left(\frac{2p+1}{\sqrt{3}}\right)+c \quad (*)$$

Assim, teremos; que (I) ficará:

$$12\int\frac{p^4\,dp}{p^3-1} = 12\int p\,dp - 12\underbrace{\int\frac{p\,dp}{p^3-1}}_{(*)} =$$

$$= \frac{\cancel{12}\,p^2}{\cancel{2}} - 12\cdot\left[\frac{1}{3}\ln|p+1| - \frac{1}{6}\ln|p^2+p+1| + \frac{1}{\sqrt{3}}\arctan\left(\frac{2p+1}{\sqrt{3}}\right)\right]+c$$

$$= 6p^2 - 4\ln|p+1| - 2\ln|p^2+p+1| + \frac{12}{\sqrt{3}}\arctan\left(\frac{2p+1}{\sqrt{3}}\right) + c$$

$$= 6\left(\sqrt[\cancel{12}\,6]{x}\right)^{\cancel{2}} - 4\ln\left|\sqrt[12]{x}+1\right| - 2\ln\left|\left(\sqrt[\cancel{12}\,6]{x}\right)^{\cancel{2}} + \sqrt[12]{x}+1\right| +$$

$$+ \frac{12}{\sqrt{3}} \cdot \arctan\left(\frac{2\sqrt[12]{x}+1}{\sqrt{3}}\right) + c$$

$x = p^{12}$

$\Rightarrow p = \sqrt[12]{x}$

$$= 6\sqrt[6]{x} + 4\ln\left|\sqrt[12]{x}+1\right| - 2\ln\left|\sqrt[6]{x} + \sqrt[12]{x}+1\right| +$$

$$+ \frac{12}{\sqrt{3}} \cdot \arctan\left(\frac{2\sqrt[12]{x}+1}{\sqrt{3}}\right) + c$$