CÁLCULO II

15/08/25 – AULA 27

LISTA 06

01 – k)

(k) $\displaystyle\int \frac{\sqrt{x}\,dx}{x^3+2x^2-3x}$

$x = p^2 \Rightarrow dx = 2p.dp$

$$\Rightarrow \int \frac{\sqrt{x}\,dx}{x^3+2x^2-3x} = \int \frac{\sqrt{\cancel{p^2}}\,.\,2p\,dp}{(p^2)^3+2(p^2)^2-3.p^2}$$

$$= \int \frac{2p^2dp}{p^6+2p^4-3p^2} = \int \frac{2\cancel{p^2}\,dp}{\cancel{p^2}(p^4+2p^2-3)} =$$

$$= \int \frac{2\,dp}{p^4+2p^2-3} =$$

$$p^4+2p^2-3=0 \Leftrightarrow p^2 = \frac{-2\pm\sqrt{4+12}}{2\cdot 1}$$

$$\Leftrightarrow p^2 = \frac{-2\pm 4}{2} \begin{cases} \nearrow p^2 = 1 \\ \searrow p^2 = -3 \end{cases}$$

$$p^4+2p^2-3 = (p^2-1).(p^2+3) = (p+1)(p-1)(p^2+3)$$

$$\textcircled{=} \int \frac{2\,dp}{(p+1)(p-1)(p^2+3)}$$

$$\frac{2}{(p+1)(p-1)(p^2+3)} = \frac{A}{p+1} + \frac{B}{p-1} + \frac{Cp+D}{p^2+3}$$

$$= \frac{A(p-1)(p^2+3) + B(p+1)(p^2+3) + (Cp+D)(p^2-1)}{(p+1)(p-1)(p^2+3)}$$

$$\Leftrightarrow 2 = Ap^3 + 3Ap - Ap^2 - 3A + Bp^3 + 3Bp + Bp^2 + 3B +$$
$$+ Cp^3 - Cp + Dp^2 - D$$

$$+\begin{cases} A + B + C = 0 \\ -A + B + D = 0 \\ 3A + 3B - C = 0 \\ -3A + 3B - D = 2 \end{cases}$$

$$8B = 2 \Rightarrow \boxed{B = \frac{1}{4}}$$

$$\begin{cases} -A + B + D = 0 & \times(-3) \\ -3A + 3B - D = 2 \end{cases}$$

$$\Downarrow$$

$$\begin{cases} 3\cancel{A} - 3\cancel{B} - 3D = 0 \\ -3\cancel{A} + 3\cancel{B} - D = 2 \end{cases}$$

$$-4D = 2$$

$$\boxed{D = -\frac{1}{2}}$$

$$-A + B + D = 0 \Rightarrow A = B + D$$

$$A = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4} \Rightarrow \boxed{A = -\frac{1}{4}}$$

$$A + B + C = 0$$

$$-\frac{1}{4} + \frac{1}{4} + C = 0 \Rightarrow \boxed{C = 0}$$

Assim, obtemos:

$$\int \frac{2\,dp}{(p+1)(p-1)(p^2+3)} = \int \left(\frac{-\frac{1}{4}}{p+1} + \frac{\frac{1}{4}}{p-1} + \frac{0 - \frac{1}{2}}{p^2+3} \right) dp$$

$$= -\frac{1}{4}\int \frac{dp}{p+1} + \frac{1}{4}\int \frac{dp}{p-1} - \frac{1}{2}\int \frac{dp}{p^2+3} \rightarrow \int \frac{du}{u^2+a^2} \quad a = \sqrt{3}$$

$$= -\frac{1}{4}\ln|p+1| + \frac{1}{4}\ln|p-1| - \frac{1}{2}\,\frac{1}{\sqrt{3}}\,\arctan\left(\frac{p}{\sqrt{3}}\right) + C$$

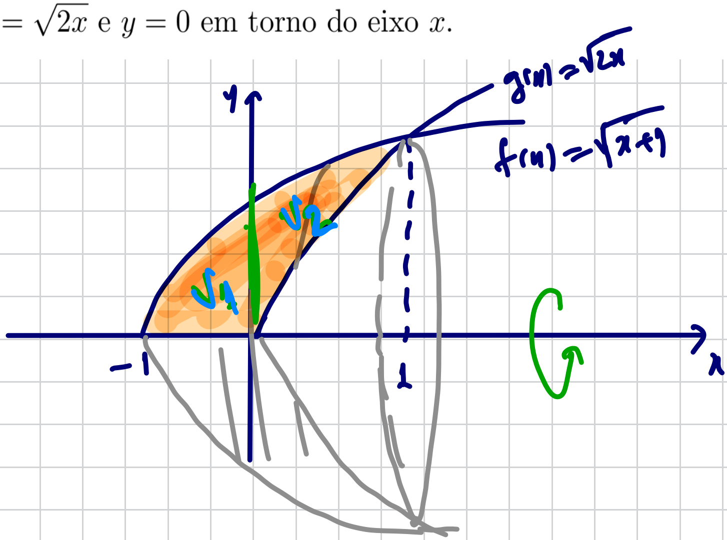
$$= -\frac{1}{4}\ln|\sqrt{x}+1| + \frac{1}{4}\ln|\sqrt{x}-1| - \frac{1}{2\sqrt{3}}\arctan\left(\sqrt{\frac{x}{3}}\right) + C$$

$p^2 = x$

$\Rightarrow p = \sqrt{x}$

L9:

3. Obtenha o volume do sólido S obtido ao se girar a região limitada por $y = \sqrt{x+1}$, $y = \sqrt{2x}$ e $y = 0$ em torno do eixo x.



V_1: volume "maciço"

V_2: volume "cavado".

$V = V_1 + V_2$

intercepto: $f(x) = g(x)$

$$(\sqrt{x+1})^2 = (\sqrt{2x})^2$$

$$x + 1 = 2x \Rightarrow \boxed{x = 1}$$

$V = V_1 + V_2$; onde:

$$\bullet\ V_1 = \pi \int_{-1}^{0} [f(x)]^2 dx = \pi \int_{-1}^{0} [\sqrt{x+1}]^{\cancel{2}} dx$$

$$= \pi \cdot \int_{-1}^{0} (x+1) dx = \pi \cdot \left.\frac{(x+1)^2}{2}\right|_{-1}^{0} =$$

$$= \pi \cdot \left(\frac{1}{2} - 0\right) = \frac{\pi}{2}\ u.v$$

$$\bullet\ V_2 = \pi \int_{0}^{1} [(f(x))^2 - (g(x))^2] dx$$

$$= \pi \int_{0}^{1} \left((\sqrt{x+1})^{\cancel{2}} - (\sqrt{2x})^{\cancel{2}}\right) dx$$

$$= \pi \cdot \int_{0}^{1} (x+1-2x) dx = \pi \int_{0}^{1} (1-x) dx =$$

$$= \pi \cdot \left.\left(x - \frac{x^2}{2}\right)\right|_{0}^{1} = \pi \cdot \left(1 - \frac{1}{2} - 0\right) = \frac{\pi}{2}$$

Portanto, obtemos:

$$V = V_1 + V_2 = \frac{\pi}{2} + \frac{\pi}{2} = \pi \text{ u.v.}$$

L9

8. Em cada item a seguir, achar o volume do sólido S obtido ao se girar a curva $y = f(x)$, em torno do eico x, no intervalo $[a, b]$ dado.

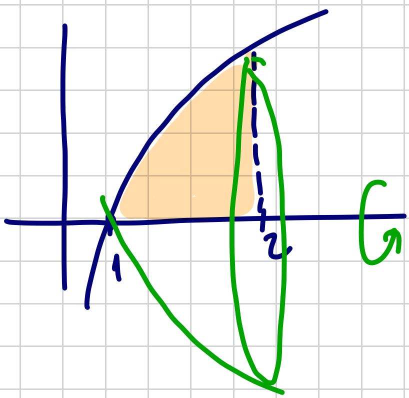
(a) $f(x) = \cos x$, $[-1, 1]$.

(b) $f(x) = \sec x$, $[0, \frac{\pi}{4}]$.

(c) $f(x) = \ln x$, $[1, 2]$.

(d) $\sqrt{\frac{\ln x}{x} + \frac{x}{\sqrt{x^2 - 1}}}$, $[e, e^2]$.

(c)



$$V = \pi \cdot \int_1^2 [f(x)]^2 \, dx =$$

$$= \pi \int_1^2 (\ln x)^2 \, dx$$

• $\int \ln^2 x \, dx = \int u \, dv = u \cdot v - \int v \, du$

$$\begin{cases} u = (\ln x)^2 \Rightarrow du = 2 \ln x \cdot \frac{1}{x} \, dx \\ dv = dx \Rightarrow v = x \end{cases}$$

$$\Rightarrow \int \ln^2 x\, dx = x.\ln^2 x - \int \cancel{x}.\frac{2\ln x\, dx}{\cancel{x}}$$

$$= x.\ln^2 x - 2\underbrace{\int \ln x\, dx}_{} =$$

$$\begin{cases} u = \ln x \Rightarrow du = \frac{dx}{x} \\ dv = dx \Rightarrow v = x \end{cases}$$

$$= x.\ln^2 x - 2.\left(x.\ln x - \int \cancel{x}.\frac{dx}{\cancel{x}}\right)$$

$$= x.\ln^2 x - 2x\ln x + 2x + C$$

Assim, obtemos:

$$V = \pi\int_1^2 \ln^2 x\, dx = \pi.\left(x\ln^2 x - 2x\ln x + 2x\right)\Big|_1^2 =$$

$$= \pi.\left(2\ln^2 2 - 2.2.\ln 2 + 4 - \left(\underbrace{1.\ln^2 1}_{=0} - 2.\underbrace{\ln 1}_{=0} + 2\right)\right)$$

$$= \pi.\left(2\ln^2 2 - 4\ln 2 + 4 - 2\right) =$$

$$= \pi\left(2\ln^2 2 - 4\ln 2 + 2\right) \text{ u.v.}$$

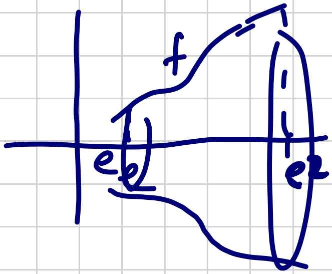
(d) $f(x) = \sqrt{\frac{\ln x}{x} + \frac{x}{\sqrt{x^2-1}}}$; $[e, e^2]$

Note que, no intervalo $[e, e^2]$;

$\ln x > 0$ (a partir de $x > 1$)

$\Rightarrow \frac{\ln x}{x} > 0$; e $\frac{x}{\sqrt{x^2-1}} > 0$

conclusão: $f(x) > 0$. em $[e, e^2]$



$$\Rightarrow V = \pi \cdot \int_e^{e^2} [f(x)]^2 \, dx =$$

$$= \pi \int_e^{e^2} \left(\cancel{\sqrt{\frac{\ln x}{x} + \frac{x}{\sqrt{x^2-1}}}}\right)^{\cancel{2}} dx = \pi \int_e^{e^2} \left(\frac{\ln x}{x} + \frac{x}{\sqrt{x^2-1}}\right) dx$$

$$= \pi \int_e^{e^2} \frac{\ln x}{x} \, dx + \pi \int_e^{e^2} \frac{x \, dx}{\sqrt{x^2-1}} \longrightarrow -\frac{1}{2} \int (x^2-1)^{-\frac{1}{2}} (-2x) \, dx$$

$u = x^2 - 1$

$du = -2x \, dx$

$\int u^k \, du$; $u = \ln x \Rightarrow du = \frac{dx}{x}$ $(k=1)$

$$= \pi \cdot \frac{(\ln x)^2}{2}\Big|_e^{e^2} + \pi \cdot \left(-\frac{1}{2}\right) \cdot \frac{(x^2-1)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\Big|_e^{e^2} =$$

$$[\ln(e \cdot e)]^2$$

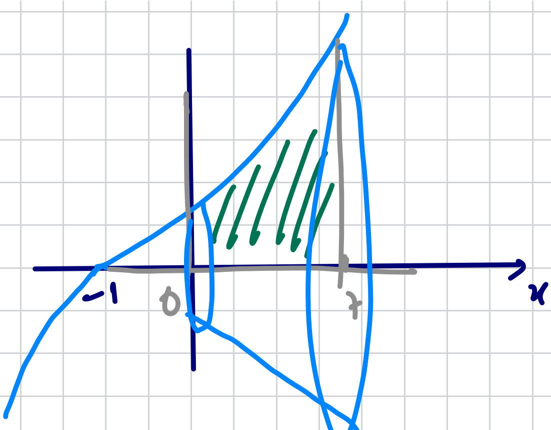
$$= \frac{\pi}{2}\left((\ln e^2)^2 - (\ln e)^2\right) - \frac{\pi}{2}\left[2\sqrt{e^4-1} - 2\sqrt{e^2-1}\right]$$

$$= \frac{\pi}{2} \cdot \left[(\ln e + \ln e)^2 - 1^2\right] - \pi\left(\sqrt{e^4-1} - \sqrt{e^2-1}\right)$$

$$\frac{3\pi}{2} - \pi\left(\sqrt{e^4-1} - \sqrt{e^2-1}\right)$$

L9)

10. Obtenha o volume do sólido obtido ao se girar a região R limitada pelas curvas $y = (x+1)^{\frac{1}{3}}$, $x = 0$, $x = 7$ e $y = 0$ em torno do eixo Ox. (Resp.: $\frac{3\pi}{5}(8^{\frac{5}{3}} - 1)$ u.v.)



$$V = \pi \int_0^7 [f(x)]^2 \, dx =$$

$$\pi \int_0^7 (x+1)^{\frac{2}{3}} \, dx$$

- $\int (x+1)^{\frac{2}{3}} dx = \frac{(x+1)^{\frac{2}{3}+1}}{\frac{2}{3}+1} + C = \frac{(x+1)^{\frac{5}{3}}}{\frac{5}{3}} + C$

$u = x+1$

$du = dx$

$$= \frac{3}{5}(x+1)^{\frac{5}{3}} + C$$

Disso, obtemos:

$$V = \pi \int_0^7 (x+1)^{\frac{2}{3}} dx = \pi \cdot \left(\frac{3}{5}(x+1)^{\frac{5}{3}} \right) \Bigg|_0^7 =$$

$$\frac{3\pi}{5} \cdot \left((7+1)^{\frac{5}{3}} - (0+1)^{\frac{5}{3}} \right) = \frac{3\pi}{5} \cdot \left(8^{\frac{5}{3}} - 1 \right)$$

L 10

11. Ache o comprimento do arco da curva $y = \frac{1}{3}\sqrt{x}(3x-1)$ do ponto onde $x = 1$ ao ponto onde $x = 4$. (Resp.: $\frac{22}{3}$)

$$l = \int_1^4 \sqrt{1 + [f'(x)]^2}\, dx$$

$$f(x) = \underbrace{\frac{1}{3} x^{1/2}}_{u} \underbrace{(3x-1)}_{v} = u \cdot v$$

$$f'(x) = u \cdot v' + u' \cdot v$$

$$f'(x) = \frac{1}{\cancel{3}} \cdot x^{1/2} \cdot \cancel{3} + \frac{1}{3} \cdot \frac{1}{2} x^{-1/2} \cdot (3x-1)$$

$$f'(x) = \sqrt{x} + \frac{1}{6} \frac{(3x-1)}{\sqrt{x}}$$

$$\Rightarrow f'(x) = \frac{6x + 3x - 1}{6\sqrt{x}} = \frac{9x-1}{6\sqrt{x}} .$$

Disso, vamos obter:

$$\ell = \int_1^4 \sqrt{1+\left(\frac{9x-1}{6\sqrt{x}}\right)^2}\,dx =$$

$$= \int_1^4 \sqrt{1+\frac{81x^2-18x+1}{36x}}\,dx =$$

$$= \int_1^4 \sqrt{\frac{36x+81x^2-18x+1}{36x}}\,dx =$$

$$= \int_1^4 \sqrt{\frac{81x^2+18x+1}{36x}}\,dx = \int_1^4 \frac{\sqrt{(9x+1)^2}}{6\sqrt{x}}\,dx =$$

$$= \frac{1}{6}\int_1^4 \frac{9x+1}{\sqrt{x}}\,dx = \frac{1}{6}\int_1^4 \left(\frac{9x}{\sqrt{x}}+\frac{1}{\sqrt{x}}\right)dx$$

$$= \frac{9}{6}\int_1^4 x\cdot x^{-\frac{1}{2}}\,dx + \frac{1}{6}\int_1^4 x^{-\frac{1}{2}}\,dx$$

$$= \frac{3}{2}\int_1^4 x^{\frac{1}{2}}\,dx + \frac{1}{6}\int_1^4 x^{-\frac{1}{2}}\,dx =$$

$$= \cancel{\frac{3}{2}} \left(\frac{x^{\frac{3}{2}}}{\cancel{\frac{3}{2}}} \right) \Bigg|_1^4 + \frac{1}{\cancel{6}} \cdot \left(\frac{x^{\frac{1}{2}}}{\frac{1}{\cancel{2}}} \right) \Bigg|_1^4 =$$

$$\left(x\sqrt{x} \right) \Big|_1^4 + \frac{1}{3} \left(\sqrt{x} \right) \Big|_1^4 =$$

$$4\sqrt{4} - 1\sqrt{1} + \frac{1}{3} \left(\sqrt{4} - \sqrt{1} \right)$$

$$= 8 - 1 + \frac{1}{3}(2-1) = 7 + \frac{1}{3} = \frac{22}{3}$$