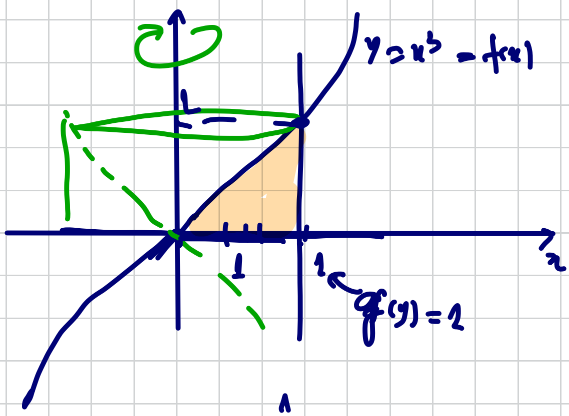
CÁLCULO II

01/08/25 – AULA 24

LISTA 09

15. Use o método do invólucro cilíndrico para determinar o volume do sólido gerado quando a região determinada pelas curvas abaixo girar em torno do eixo y.

(a) $y = x^3$, $x = 1$, $y = 0$.



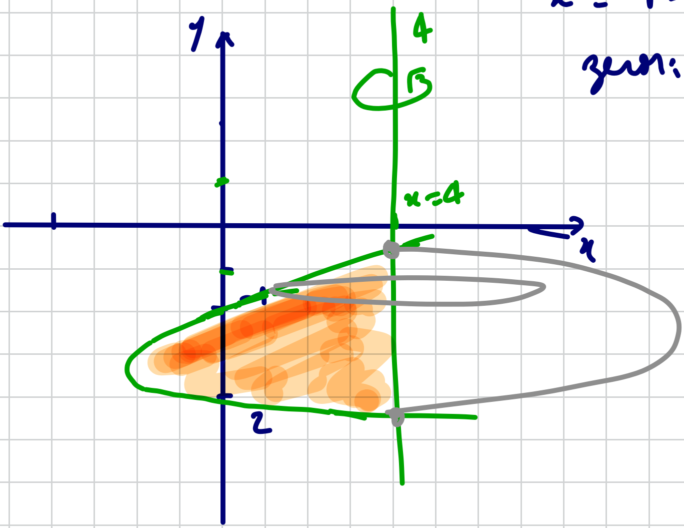
$$V = \pi \cdot (1)^2 \cdot 1 - 2\pi \int_0^1 x \cdot f(x)\, dx$$

$$= \pi - 2\pi \int_0^1 x \cdot x^3\, dx = \pi - 2\pi \int_0^1 x^4\, dx$$

$$= \pi - 2\pi \cdot \frac{x^5}{5}\Big|_0^1 = \pi - \frac{2\pi}{5} = \frac{3\pi}{5} \text{ u.v.}$$

Um outro exercício:

13. Ache o volume do sólido gerado pela rotação, em torno da reta $x = 4$, da região limitada por aquela reta e pela parábola $x = 4 + 6y + 2y^2$. (Resp.: $\frac{1250}{3}\pi$ u.v.)



$x = 4 + 6y + 2y^2$

zero: $4 + 6y + 2y^2 = 0$

$y^2 + 3y + 2 = 0$

$y = \dfrac{-3 \pm \sqrt{9-8}}{2}$

$y = \dfrac{-3 \pm 1}{2}$ $\rightarrow y = -1$, $y = -2$

$+ \quad - \quad +$

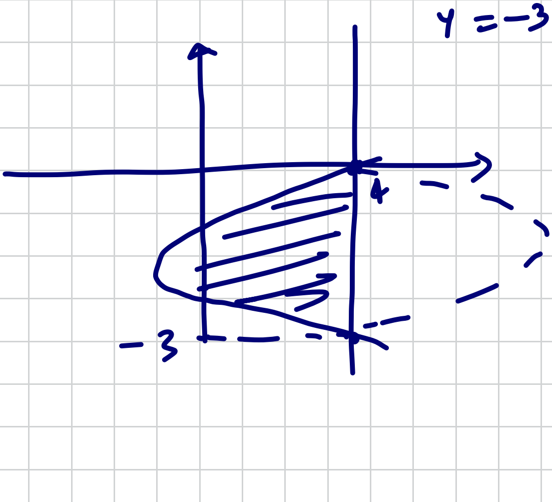
$-2 \quad -1$

interceptos: $\cancel{4} + 6y + 2y^2 = \cancel{4}$

$2y(3+y) = 0$

$y = 0$

$y = -3$



Usaremos o método de disco.

$$V = \pi \cdot \int_{-3}^{0} (f(y) - 4)^2 \, dy$$

$$= \pi \cdot \int_{-3}^{0} (\cancel{4} + 6y + 2y^2 - \cancel{4})^2 \, dy$$

$$= \pi \int_{-3}^{0} (6y + 2y^2)^2 \, dy =$$

$$= \pi \int_{-3}^{0} (36y^2 + 24y^3 + 4y^4) \, dy =$$

$$= \pi \cdot \left[\frac{36y^3}{3} + \frac{24y^4}{4} + \frac{4y^5}{5} \right] \Bigg|_{-3}^{0} =$$

$$= \pi \cdot \left(12y^3 + 6y^4 + \frac{4y^5}{5} \right) \Bigg|_{-3}^{0} =$$

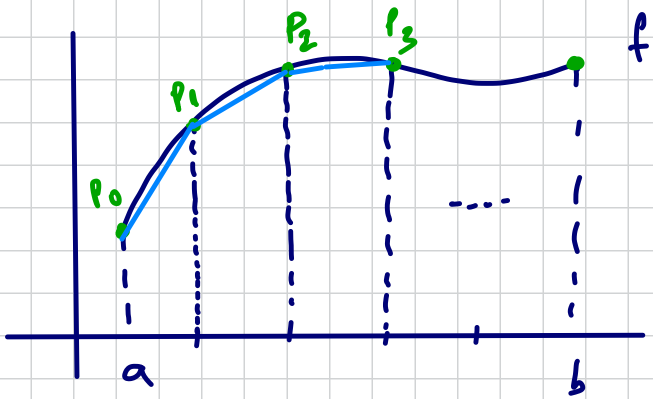
$$= \pi \cdot \left(0 - \left(12 \cdot (-3)^3 + 6 \cdot (-3)^4 + \frac{4 \cdot (-3)^5}{5} \right) \right) \text{ u.a.}$$

$(\ldots)$

CÁLCULO DE COMPRIMENTO DE CURVA.

Seja $f: [a,b] \to \mathbb{R}$ uma função derivável em (a,b). Queremos obter o comprimento da curva descrita pelo gráfico de f no intervalo $[a,b]$.

Seja $P = \{a = t_0 < t_1 < t_2 < \ldots < t_n = b\}$ uma partição qualquer do intervalo $[a,b]$, determinando pontos $P_i(t_i, f(t_i))$ sobre o gráfico de f.

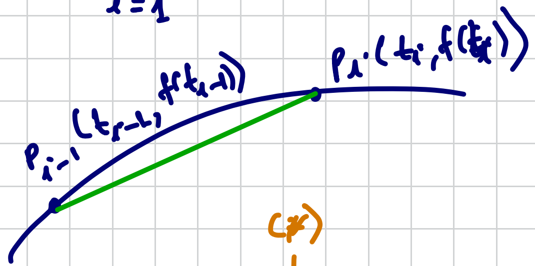


$$P_0(a, f(a));\quad P_1(t_1, f(t_1));\quad \ldots;\ P_{i}(t_{i}, f(t_{i}))$$

A ideia para determinar a medida do comprimento l do gráfico de f em $[a,b]$ é ir aproximando pelo comprimento $\tilde{\ell}$ dado pela soma dos comprimentos dos segmentos $\overline{P_{i-1}P_i}$ que determinam a linha poligonal.

$$\tilde{\ell} = \overline{P_0P_1} + \overline{P_1P_2} + \overline{P_2P_3} + \dots + \overline{P_{n-1}P_n} =$$

$$= \sum_{i=1}^{n} \overline{P_{i-1}P_i} \quad ; \quad \text{onde:}$$

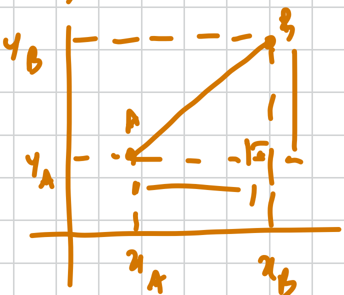


$$\overline{P_{i-1}P_i} = d(P_{i-1}, P_i) \overset{(*)}{=} \sqrt{(t_i - t_{i-1})^2 + (f(t_i) - f(t_{i-1}))^2}$$

Além disso, como f é derivável em (a,b); pelo T.V.M., segue que $\exists\ c_i \in [t_{i-1}, t_i]$ tal que

$$f(t_i) - f(t_{i-1}) = f'(c_i).(t_i - t_{i-1})$$

(*)



$$d_{AB} = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$$

Assim, podemos escrever:

$$\overline{P_{i-1}P_i} = \sqrt{(t_i - t_{i-1})^2 + \left(f(t_i) - f(t_{i-1})\right)^2} =$$

$$= \sqrt{\underbrace{(t_i - t_{i-1})}_{\Delta t_i}^2 + \left(f'(c_i)\cdot\underbrace{(t_i - t_{i-1})}_{\Delta t_i}\right)^2} =$$

$$= \sqrt{(\Delta t_i)^2 + f'(c_i)^2 \cdot \Delta t_i^2} = \sqrt{\Delta t_i^{\cancel{2}} \cdot \left(1 + [f'(c_i)]^2\right)}$$

$$= \sqrt{1 + [f'(c_i)]^2} \cdot \Delta t_i$$

Disso, tem-se:

$$\tilde{l} = \sum_{i=1}^{n} \overline{P_{i-1}P_i} = \sum_{i=1}^{n} \sqrt{1 + [f'(c_i)]^2} \cdot \Delta t_i \, ;$$

uma soma de Riemann.

Então, a medida do comprimento l será:

$$l = \lim_{n\to\infty} \tilde{l} = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + [f'(c_i)]^2} \cdot \Delta t_i =$$

$$= \int_a^b \sqrt{1 + [f'(x)]^2} \, dx$$

conclusão:

$$\boxed{l = \int_a^b \sqrt{1+[f'(x)]^2}\,.dx}$$

EXEMPLOS: LISTA 10:

1. Calcular o comprimento de arco de cada curva dada no intervalo considerado em cada caso:

   (a) $f(x) = 2\sqrt{x}$, em ~~[0, 1]~~. $[1,2]$.

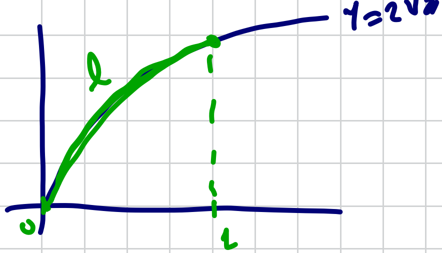
   (b) $f(x) = \ln\cos x$, em $[0, \frac{\pi}{3}]$.

   (c) $f(x) = x^{\frac{3}{2}}$, em $[0, 1]$.

   (d) $f(x) = \ln\dfrac{e^x+1}{e^x-1}$, onde $0 \leq a \leq x \leq b$.

(a) $f(x) = 2x^{\frac{1}{2}} \Rightarrow \underbrace{f'(x)}_{} = \cancel{2}\cdot\frac{1}{\cancel{2}}x^{-\frac{1}{2}} = x^{-\frac{1}{2}} = \underbrace{\frac{1}{\sqrt{x}}}_{}$.

$$\Rightarrow l = \int_1^2 \sqrt{1+[f'(x)]^2}\,dx = \int_1^2 \sqrt{1+\left(\frac{1}{\cancel{\sqrt{x}}}\right)^{\cancel{2}}}\,dx$$



$$= \int_1^2 \sqrt{1+\frac{1}{x}}\,dx$$

$$= \int_1^2 \sqrt{\frac{x+1}{x}}\,dx$$

$\underline{\underline{1^o}}$: $\int \sqrt{\frac{1+x}{x}}\, dx = ?$

$$\frac{1+x}{x} = p^2 \rightsquigarrow p = \sqrt{\frac{1+x}{x}}$$

$$1+x = p^2 . x$$

$$p^2 x - x = 1$$

$$x(p^2-1) = 1$$

$$x = \frac{1}{p^2-1} = (p^2-1)^{-1}$$

$$\Rightarrow dx = -1.(p^2-1)^{-2} . (2p\, dp)$$

$$\Rightarrow dx = -\frac{2p\, dp}{(p^2-1)^2}$$

$$\Rightarrow \int \sqrt{\frac{1+x}{x}}\, dx = \int \sqrt{p^{\cancel{2}}} . \left(-\frac{2p}{(p^2-1)^2}\right) dp$$

$$= \int \frac{-2p^2}{(p^2-1)^2}\, dp = \int \frac{-2p^2\, dp}{(p+1)^2 (p-1)^2} =$$

$$\frac{-2p^2}{(p+1)^2(p-1)^2} = \frac{A}{(p+1)^2} + \frac{B}{p+1} + \frac{C}{(p-1)^2} + \frac{D}{p-1}$$

$$= \frac{A(p-1)^2 + B(p+1)(p-1)^2 + C(p+1)^2 + D.(p-1)(p+1)^2}{(p+1)^2(p-1)^2}$$

$$-2p^2 = Ap^2 - 2Ap + A + (Bp+B).(p^2-2p+1) + Cp^2 + 2Cp + C + (Dp-D).(p^2+2p+1)$$

$$-2p^2 = Ap^2 - 2Ap + A + Bp^3 - 2Bp^2 + Bp + Bp^2 - 2Bp + B + Cp^2 + 2Cp + C + Dp^3 + 2Dp^2 + Dp - Dp^2 - 2Dp - D$$

$$+\begin{cases} B + D = 0 \\ A - B + C + D = -2 \quad (\bullet) \\ -2A - B + 2C - D = 0 \\ A + B + C - D = 0 \quad (\bullet\bullet) \end{cases}$$

$$4C = -2 \Rightarrow C = -\frac{1}{2}$$

$(\bullet) + (\bullet\bullet)$: $2A + 2C = -2$

$$2A + 2.\left(-\frac{1}{2}\right) = -2$$

$$2A - 1 = -2$$

$$A = -\frac{1}{2}$$

De (••), teremos: $A + B + C - D = 0$

$$-\frac{1}{2} + B - \frac{1}{2} - D = 0$$

$$\boxed{B - D = 1}$$

Juntando com a 1.ª eq., vem:

$$+\begin{cases} B + D = 0 \\ B - D = 1 \end{cases}$$

$$2B = 1 \Rightarrow B = \frac{1}{2} \Rightarrow D = -\frac{1}{2}$$

$$= \int \left(\frac{A}{(p+1)^2} + \frac{B}{p+1} + \frac{C}{(p-1)^2} + \frac{D}{p-1} \right) dp =$$

$$= -\frac{1}{2} \int \frac{dp}{(p+1)^2} + \frac{1}{2} \int \frac{dp}{p+1} - \frac{1}{2} \int \frac{dp}{(p-1)^2} - \frac{1}{2} \int \frac{dp}{p-1}$$

$$= -\frac{1}{2} \int (p+1)^{-2} dp + \frac{1}{2} \ln|p+1| - \frac{1}{2} \int (p-1)^{-2} dp - \frac{1}{2} \ln|p-1| + C$$

$$= -\frac{1}{2} \frac{(p+1)^{-1}}{-1} + \frac{1}{2} \ln|p+1| - \frac{1}{2} \frac{(p-1)^{-1}}{-1} - \frac{1}{2} \ln|p-1| + C$$

$$= \frac{1}{2} \cdot \left[+ \frac{1}{p+1} + \frac{1}{p-1} + \ln \left| \frac{p+1}{p-1} \right| \right] + C =$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{\frac{x+1}{x}}+1}+\frac{1}{\sqrt{\frac{x+1}{x}}-1}+\ln\left|\frac{\sqrt{\frac{x+1}{x}}+1}{\sqrt{\frac{x+1}{x}}-1}\right|\right]+C$$

$$\Rightarrow l = \int_1^2 \sqrt{\frac{1+x}{x}}\,dx =$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{\frac{3}{2}}+1}+\frac{1}{\sqrt{\frac{3}{2}}-1}+\ln\left|\frac{\sqrt{\frac{3}{2}}+1}{\sqrt{\frac{3}{2}}-1}\right|\right] -$$

$$- \frac{1}{2}\left[\frac{1}{\sqrt{2}+1}+\frac{1}{\sqrt{2}-1}+\ln\left|\frac{\sqrt{2}+1}{\sqrt{2}-1}\right|\right]$$

(b) $f(x) = \ln \cos x \; ; \qquad [0, \frac{\pi}{3}].$

$$f'(x) = \frac{-\operatorname{sen} x}{\cos x} = -\tan x .$$

$$\boxed{(\ln u)' = \frac{u'}{u}}$$

$$\Rightarrow l = \int_0^{\frac{\pi}{3}} \sqrt{1 + [f'(x)]^2}\, dx = \int_0^{\frac{\pi}{3}} \sqrt{1 + (-\tan x)^2}\, dx$$

$$= \int_0^{\frac{\pi}{3}} \sqrt{1 + \tan^2 x}\, dx = \int_0^{\frac{\pi}{3}} \sqrt{\sec^{\cancel{2}} x}\, dx =$$

$1 + \tan^2 x = \sec^2 x$

$$= \int_0^{\frac{\pi}{3}} \sec x\, dx = \ln |\sec x + \tan x| \Big|_0^{\frac{\pi}{3}} =$$

$\int \sec u\, du = \ln |\sec u + \tan u| + c$

$$= \ln \left| \sec \frac{\pi}{3} + \tan \frac{\pi}{3} \right| - \ln |\sec 0 + \tan 0|$$

$$= \ln|2+\sqrt{3}| - \ln|1+0|$$

$$= \ln(2+\sqrt{3}) - \underbrace{\ln 1}_{=0} = \ln(2+\sqrt{3})$$

Fim da disciplina!

Entregar na quarta: LISTA 09 – nº 6.