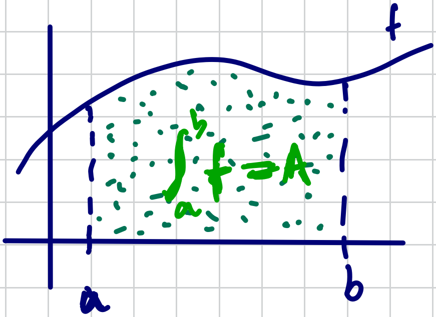
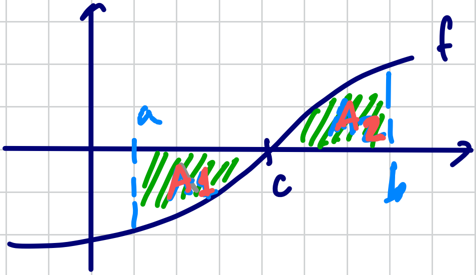
CÁLCULO II

23/07/25 – AULA 21

CÁLCULO DE ÁREAS Seja $f: [a,b] \to \mathbb{R}$ uma função integrável. Vimos que, se $f \geq 0$ em $[a,b]$, $\int_a^b f(x)dx$ fornece a medida da área abaixo do seu gráfico no intervalo $[a,b]$.

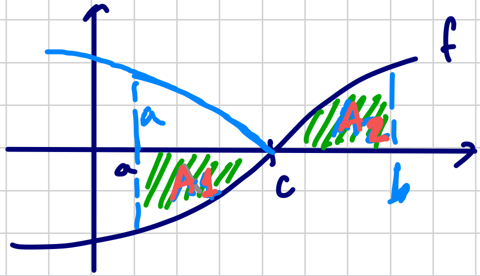


E se $f < 0$? neste caso, $\int_a^b f$ não representa uma área, mas mesmo assim, podemos obter sua área com a integral, desde que adaptada com auxílio de MÓDULO:



a área A será $A = A_1 + A_2$, onde

$$A_1 = \int_a^c |f| \quad e \quad A_2 = \int_c^b f$$

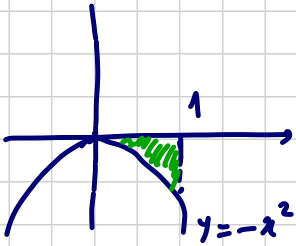


Ou seja, $A = -A_1 + A_2$; onde

$$A_1 = \int_a^c f \quad e \quad A_2 = \int_c^b f$$

Ex: $f(x) = -x^2$. Achar a área compreendida pelo gráfico de f e o intervalo $[0,1]$.

Sol.:



$$A = \int_0^1 |-x^2|\, dx \quad \text{ou}$$

$$A = -\int_0^1 -x^2\, dx$$

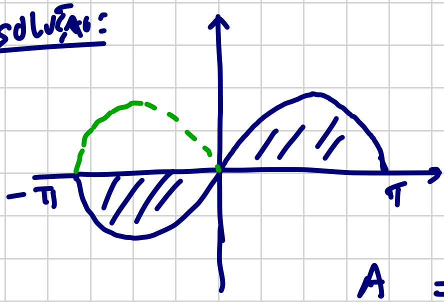
$$= \int_0^1 x^2\, dx =$$

$$= \frac{x^3}{3}\Big|_0^1 = \frac{1}{3} - 0$$

$$= \frac{1}{3} \text{ u.a.}$$

Ex: Área de $y = \operatorname{sen} x$ no intervalo $[-\pi, \pi]$

Solução:



Por simetria

$$A = 2 \cdot \int_0^{\pi} \operatorname{sen} x \, dx \quad ; \text{ ou}$$

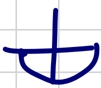
$$A = \int_{-\pi}^{0} -\operatorname{sen} x \, dx + \int_0^{\pi} \operatorname{sen} x \, dx$$

$$\boxed{\int \operatorname{sen} x \, dx = -\cos x + c}$$

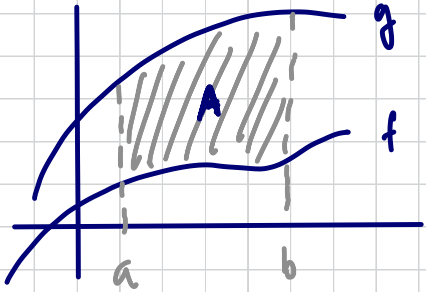
$$= +\cos x \Big|_{-\pi}^{0} + (-\cos x)\Big|_0^{\pi}$$

$$= \cos 0 - \cos(-\pi) + (-\cos \pi + \cos 0)$$

$$= 1 - (-1) + (-(-1) + 1) = 4$$

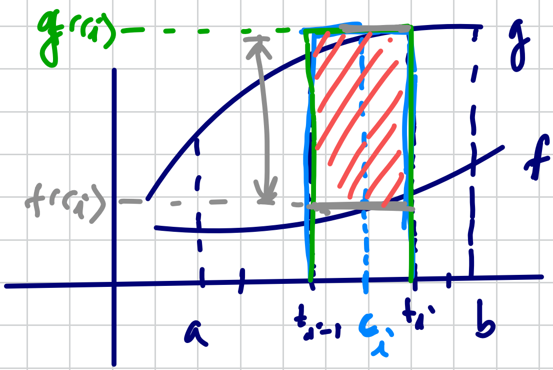


ÁREA ENTRE CURVAS. Se $f, g : [a,b] \to \mathbb{R}$ integráveis, com $f \leq g$; a área entre os gráficos de f e de g será dada por:



$$A = \int_a^b g(x)\,dx - \int_a^b f(x)\,dx = \int_a^b (g(x) - f(x))\,dx.$$

Sejamos mais precisos: dados $f, g : [a,b] \to \mathbb{R}$ integráveis, com $f \leq g$; sendo $P = \{a = t_0 < t_1 < t_2 < \dots < t_n = b\}$ uma partição do intervalo $[a,b]$ e sendo $c_i \in [t_{i-1}, t_i]$; tem-se:



A medida da área do i-ésimo retângulo será dada por

$$A_i = \underbrace{(t_i - t_{i-1})}_{\Delta t_i} . (g(c_i) - f(c_i))$$

$$\Rightarrow A_i = [g(c_i) - f(c_i)] . \Delta t_i$$

Somando sobre todos os n retângulos, obtemos a área aproximada $\tilde{A}$, dada por:

$$\tilde{A} = \sum_{i=1}^{n} [g(c_i) - f(c_i)] \cdot \Delta t_i \, .$$

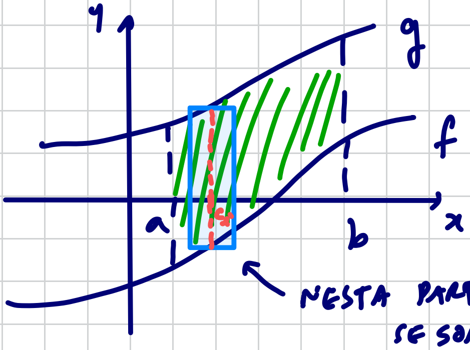
(SOMA DE RIEMANN)

A a área A será o limite da soma de Riemann acima, dada por:

$$A = \lim_{n \to \infty} \tilde{A} = \lim_{n \to \infty} \sum_{i=1}^{n} [g(c_i) - f(c_i)] \Delta t_i =$$

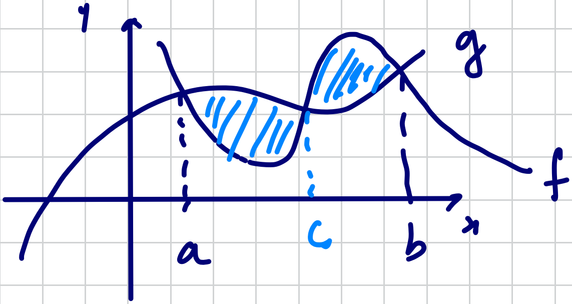
$$= \int_a^b (g(x) - f(x)) \cdot dx$$

E se algum dos gráfico ficar abaixo do eixo ox?
O resultado acima continua válido, pois:



NESTA PARTE $g(x) - f(x)$ SE SOMA.

E se os gráficos se entrelaçam? Deveremos integrar em intervalos. Por exemplo:



$$A = \int_a^c (g(x) - f(x))\,dx + \int_c^b (f(x) - g(x))\,dx.$$

Ex.: Encontre a área entre os gráficos de

$y = x^2$ e $y = 2x + 3$

Solução:

interceptos: $x^2 = 2x + 3$

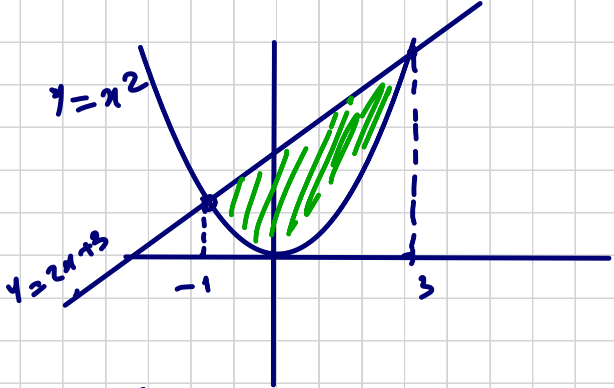
$$\Leftrightarrow x^2 - 2x - 3 = 0$$

$$x = \frac{2 \pm \sqrt{4 + 12}}{2 \cdot 1}$$

$$x = \frac{2 \pm 4}{2}$$

$\nearrow x = 3$

$\searrow x = -1$



$$A = \int_{-1}^{3} (2x+3-x^2)\,dx = \left(x^2 + 3x - \frac{x^3}{3} \right) \Big|_{-1}^{3}$$

$$= 9 + \cancel{9} - \cancel{9} - \left(1 - 3 + \frac{1}{3} \right)$$

$$= 9 - \left(-2 + \frac{1}{3} \right) = 9 + 2 - \frac{1}{3} = 11 - \frac{1}{3}$$

$$= \frac{32}{3} \text{ u.a.}$$

LISTA 08.

3. EsboÃ§ar a região entre as curvas e ache as áreas compreendidas: ç

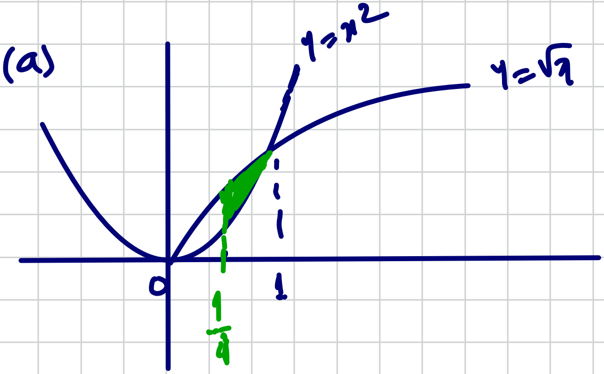
(a) $y = x^2$, $y = \sqrt{x}$, $x = \frac{1}{4}$, $x = 1$.

(b) $y = x^3 - 4x$, $y = 0$, $x = 0$, $x = 2$.

(c) $x = \operatorname{sen} y$, $x = 0$, $y = \frac{\pi}{4}$, $y = \frac{3\pi}{4}$.

(d) $y = e^x$, $y = e^{2x}$, $x = 0$, $x = \ln 2$.

(a)



interceptions :

$$x^2=\sqrt{x} \Leftrightarrow$$

$$x^4=x$$

$$\Leftrightarrow x^4-x=0$$

$$\Leftrightarrow x(x^3-1)=0$$

$$x=0$$

$$x=1$$

$$A=\int_{\frac{1}{4}}^{1}(\sqrt{x}-x^2)\,dx=\int_{\frac{1}{4}}^{1}x^{\frac{1}{2}}\,dx-\int_{\frac{1}{4}}^{1}x^2\,dx$$

$$=\left.\frac{x^{3/2}}{\frac{3}{2}}\right|_{\frac{1}{4}}^{1}-\left.\frac{x^3}{3}\right|_{\frac{1}{4}}^{1}=$$

$$=\left.\frac{2}{3}\cdot\sqrt{x^3}\right|_{\frac{1}{4}}^{1}-\left.\frac{1}{3}x^3\right|_{\frac{1}{4}}^{1}=$$

$$=\frac{2}{3}\left(\sqrt{1^3}-\sqrt{\left(\frac{1}{4}\right)^3}\right)-\frac{1}{3}\cdot\left((1)^3-\left(\frac{1}{4}\right)^3\right)$$

$$=\frac{2}{3}\left(1-\sqrt{\cancel{\left(\frac{1}{4}\right)^2}\cdot\frac{1}{4}}\right)-\frac{1}{3}\left(1-\frac{1}{64}\right)$$

$$= \frac{2}{3}\left(1 - \frac{1}{4}\cdot\frac{1}{2}\right) - \frac{1}{3}\left(\frac{63}{64}\right) =$$

$$= \frac{\cancel{2}}{3}\cdot\left(\frac{7}{\cancelto{4}{8}}\right) - \frac{1}{\cancel{3}}\cdot\left(\frac{\cancelto{21}{63}}{64}\right) = \frac{7}{12} - \frac{21}{64} =$$

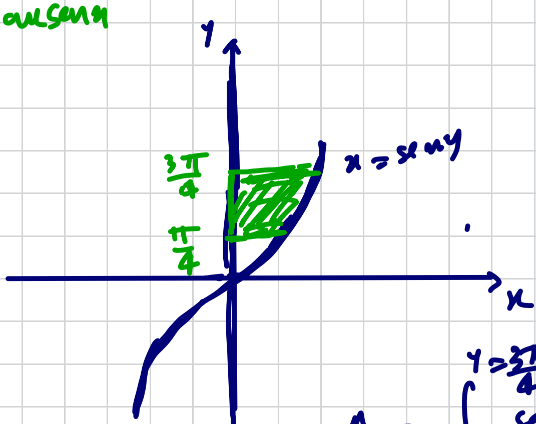
$$= \frac{112 - 63}{192} = \underline{\underline{\frac{49}{192}\ u.a.}}$$

64	12	2
32	6	2
16	3	2
8	3	2
4	3	2
2	3	2
1	3	3

(c) $x = \operatorname{sen} y$; $\underbrace{x = 0}_{\text{eixo } y}$; $y = \frac{\pi}{4}$; $y = \frac{3\pi}{4}$

$\Updownarrow$

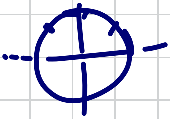
$y = \operatorname{arcsen} x$



TROCAMOS O SENTIDO DOS EIXOS PARA O CÁLCULO

$$A = \int_{y=\frac{\pi}{4}}^{y=\frac{3\pi}{4}} \operatorname{sen} y \,.\, dy =$$

$$= -\cos y \Big|_{\frac{\pi}{4}}^{\frac{3\pi}{4}} =$$



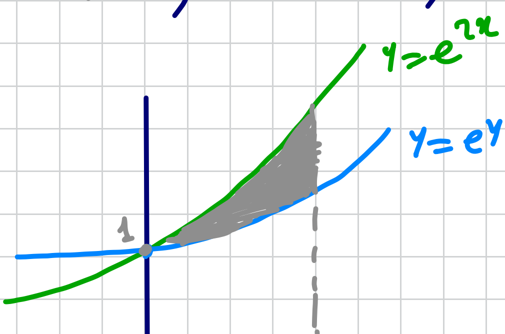
$$= -\cos\frac{3\pi}{4} - \left(-\cos\frac{\pi}{4}\right) =$$

$$= -\left(-\cos\left(\pi - \frac{3\pi}{4}\right)\right) + \cos\frac{\pi}{4}$$

$$= +\cos\frac{\pi}{4} + \cos\frac{\pi}{4} = 2\cdot\cos\frac{\pi}{4} = \cancel{2}\frac{\sqrt{2}}{\cancel{2}} = \sqrt{2} \text{ u.a.}$$

---

c) $y = e^x$; $y = e^{2x}$; $x = 0$; $x = \ln 2$



$$A = \int_0^{\ln 2} (e^{2x} - e^x)\,dx = \frac{1}{2}\int_0^{\ln 2} e^{2x} 2\,dx) - \int_0^{\ln 2} e^x\,dx$$

$$= \frac{1}{2} e^{2x}\Big|_0^{\ln 2} - e^x\Big|_0^{\ln 2} =$$

$$= \frac{1}{2}\cdot\left(e^{2\ln 2} - e^{0}\right) - \left(e^{\ln 2} - e^{0}\right)$$

$$= \frac{1}{2}\left(e^{\ln 4} - 1\right) - \left(e^{\ln 2} - 1\right)$$

$$= \frac{1}{2}(4-1) - (2-1) = \frac{1}{2}\cdot(3) - 1$$

$$= \frac{3}{2} - 1 = \frac{1}{2} \text{ u.a.}$$

obs: $\boxed{e^{\ln w} = e^{\log_e w} = w}$

L8 -

9. Calcule a área da região limitada pelas curvas $y = x^3 - 6x$ e $y = 8 - 3x^2$.

solução: interseção: $x^3 - 6x = 8 - 3x^2$

$$x^3 + 3x^2 - 6x - 8 = 0$$

$\pm 1; \pm 2; \pm 4; \pm 8$ divisores de -8

-1 é uma raiz

$$x^3 + 3x^2 - 6x - 8 \;\; \underline{\big|\, x - (-1)}$$

$$\begin{array}{l|l} x^3+3x^2-6x-8 & x+1 \\ \hline -x^3-x^2 & x^2+2x-8 \end{array}$$

$2x^2-6x-8$

$-2x^2-2x$

$-8x-8$

$+8x+8$

0

$x^2+2x-8=0$

$$x=\frac{-2\pm\sqrt{4+32}}{2}$$

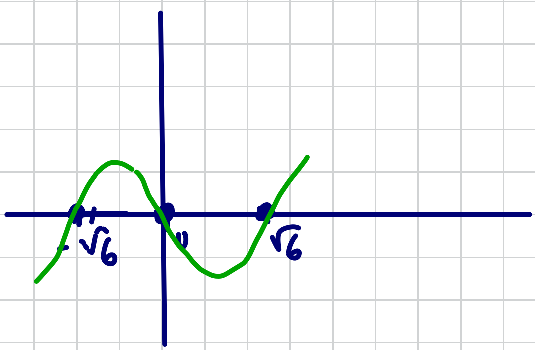
$$x=\frac{-2\pm6}{2}$$

$x=2$

ou

$x=-4$

raízes: -1; -4; $+2$. (interceptos dos dois gráficos)



zeros para

$y=x^3-6x$

$x^3-6x=0$

$x(x^2-6)=0$

$x=0$

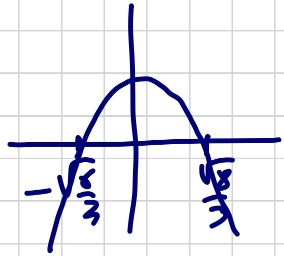
$x=\pm\sqrt{6}$

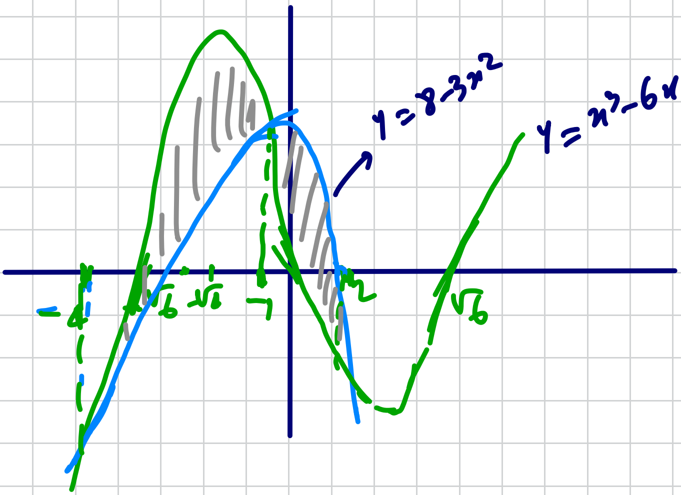
zeros para $y=8-3x^2$:

$$8-3x^2=0$$

$$3x^2=8$$

$$x=\pm\sqrt{\frac{8}{3}}$$





$$A=\int_{-4}^{-1}\left(x^3-6x-\left(8-3x^2\right)\right)dx+\int_{-1}^{2}\left(8-3x^2-\left(x^3-6x\right)\right)dx$$

$$=\int_{-4}^{-1}\left(x^3+3x^2-6x-8\right)dx+\int_{-1}^{2}\left(-x^3-3x^2+6x+8\right)dx$$

$$= \text{etc} \ldots$$