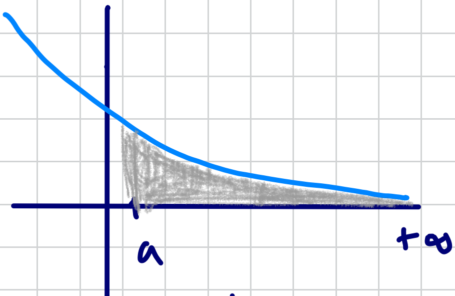
CÁLCULO II

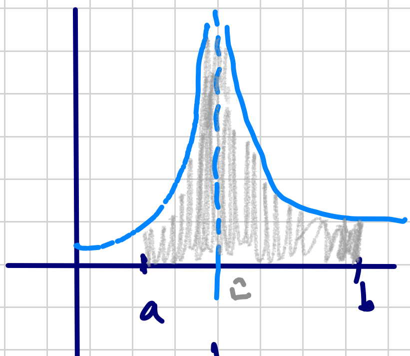
16/07/25 — AULA 19

INTEGRAIS IMPRÓPRIAS: São integrais cujo cálculo, em certos cálculos pode não existir, ou resultar em infinito. São dois tipos de integrais impróprias:

1º TIPO: quando o limite de integração é ilimitado.

2º TIPO: quando $\int_a^b f$ é tal que f não está definida em algum ponto dentro do intervalo de integração.





$$\exists \int_a^{+\infty} f(x)\,dx = ?$$

$$\exists \int_a^{b} f(x)\,dx = ?$$

Vejamos como calcular cada tipo.

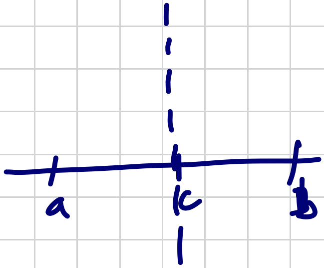
1º: quando o limite de integração é infinito.

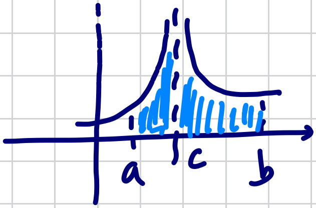
Neste caso, faz-se:

$$\int_a^{+\infty} f(x)dx = \lim_{b\to+\infty} \int_a^b f(x)dx .$$

Se existir este limite, dizemos que a integral é convergente e converge para o valor encontrado. Do contrário, é dita divergente.

2º: $\int_a^b f(x)dx$, sendo que $c \in [a,b]$ é tal que $\nexists f(c)$.



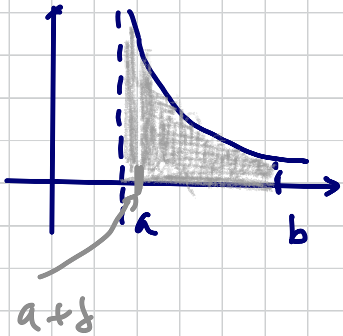


Neste caso, tome $\delta_1, \delta_2 > 0$ pequenos, e faz-se:

$$\int_a^b f(x)dx = \lim_{\delta_1\to 0^+} \int_a^{c-\delta_1} f(x)dx + \lim_{\delta_2\to 0^+} \int_{c+\delta_2}^{b} f(x)dx .$$

Se existir o limite, a integral é dita convergente, e se não existir, divergente.

O caso mais simples é quando f não está definida em apenas uma das extremidades:



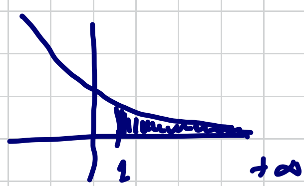
Tome $\delta > 0$ e calcule:

$$\int_a^b f(x)dx = \lim_{\delta \to 0^+} \int_{a+\delta}^{b} f(x)dx$$

Vejamos exemplos:

01) $\displaystyle\int_1^{+\infty} e^{-x}dx = ?$

$$f(x) = e^{-x} = \frac{1}{e^x} = \left(\frac{1}{e}\right)^x$$



A antiderivada será:

$$\int e^{-x}\,dx = -\int e^{-x}\cdot\underbrace{(-dx)}_{du} =$$

$$\int e^{u}\,du \qquad u=-x \qquad du=-dx$$

$$= -e^{-x} + C = -\frac{1}{e^{x}} + C.$$

Assim, teremos:

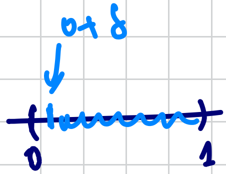
$$\int_1^{+\infty} e^{-x}\,dx = \lim_{b\to+\infty}\int_1^{b} e^{-x}\,dx = \lim_{b\to+\infty}\left(-\frac{1}{e^{x}}\right)\Bigg|_1^{b} =$$

$$= \lim_{b\to+\infty}\left(-\frac{1}{e^{b}} + \frac{1}{e^{1}}\right) = \underbrace{-\frac{1}{e^{+\infty}}}_{0} + \frac{1}{e^{1}} = \frac{1}{e}$$

conclusão: $\displaystyle\int_1^{+\infty} e^{-x}\,dx = \frac{1}{e}$

02) $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}} = ?$

$f(x) = \dfrac{1}{\sqrt{x}}$ ↳ não está definida em $x=0$



A antiderivada será:

$$\int \frac{dx}{\sqrt{x}} = \int x^{-1/2}dx = \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C$$

$$= 2\sqrt{x} + C.$$

Disso, tem-se, tomando $\delta > 0$ pequeno:

$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{\delta \to 0^+} \int_{0+\delta}^1 \frac{dx}{\sqrt{x}} = \lim_{\delta \to 0^+} (2\sqrt{x})\Big|_\delta^1 =$$

$$\lim_{\delta \to 0^+} (2 - \underbrace{2\sqrt{\delta}}_{\to 0}) = \underline{\underline{2}}$$

03) $$\int_{-\infty}^{+\infty} \frac{dx}{x^2+4x+5} = ?$$

Zeros:

$x^2+4x+5 = 0$

$\Rightarrow x = \frac{-4 \pm \sqrt{16-20}}{2} \notin \mathbb{R}$

logo, $f(x) = \frac{1}{x^2+4x+5}$

está definida em toda a reta real.

$$\int \frac{du}{u^2+a^2} = \frac{1}{a}\arctan\left(\frac{u}{a}\right) + C$$

A antiderivada de f será:

$$\int \frac{dx}{x^2+4x+5} = \int \frac{dx}{(x+2)^2+1} =$$

$$= \frac{1}{1} \cdot \arctan\left(\frac{x+2}{1}\right) + c = \arctan(x+2)+c$$

Assim, teremos:

$$\int_{-\infty}^{+\infty} \frac{dx}{x^2+4x+2} = \int_{-\infty}^{0} \frac{dx}{x^2+4x+2} + \int_{0}^{+\infty} \frac{dx}{x^2+4x+2} =$$

$$= \lim_{a \to -\infty} \int_{a}^{0} \frac{dx}{x^2+4x+2} + \lim_{b \to +\infty} \int_{0}^{b} \frac{dx}{x^2+4x+2} =$$

$$= \lim_{a \to -\infty} \arctan(x+2)\Big|_{a}^{0} + \lim_{b \to +\infty} \arctan(x+2)\Big|_{0}^{b} =$$

$$= \lim_{a \to -\infty} \left[\arctan(2) - \arctan(a+2)\right] +$$

$$+ \lim_{b \to +\infty} \left[\arctan(b+2) - \arctan(2)\right]$$

$$= \cancel{\arctan 2} - \left(\arctan(-\infty)\right) + \arctan(+\infty) - \cancel{\arctan 2}$$

$$= -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2} = \pi$$

04) $\displaystyle\int_0^1 \frac{dx}{x-1} = ?$

Tome $\delta > 0$ e calcule $\quad 1-\delta \,;\ \mid\ \delta > 0$

$$\int_0^1 \frac{dx}{x-1} = \lim_{\delta \to 0^+} \int_0^{1-\delta} \frac{dx}{x-1} \ ; \text{ onde}$$

$$\int \frac{dx}{x-1} = \ln|x-1| + C \ ; \text{ e disso:}$$

$$\int_0^1 \frac{dx}{x-1} = \lim_{\delta \to 0^+} \ln|x-1| \Big|_0^{1-\delta} =$$

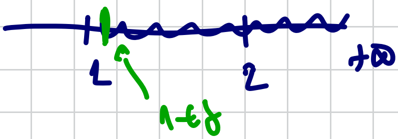
$$= \lim_{\delta \to 0^+} \ln|\cancel{1} - \delta - \cancel{1}| - \ln|0-1|$$

$$= \lim_{\delta \to 0^+} \ln|-\delta| - \underbrace{\ln 1}_{=0} =$$

$$= \lim_{\delta \to 0^+} \ln|\delta| = +\infty$$; ou seja, esta integral diverge.

05) $$\int_1^{+\infty} \frac{dx}{\sqrt{x^2-x}}$$

$$\frac{1}{\sqrt{x^2-x}}$$ não tem sentido em 0 e em 1. (no caso, em 1 é mesmo interesse)



Então, $$\int_1^{+\infty} \frac{dx}{\sqrt{x^2-x}} = \int_1^2 \frac{dx}{\sqrt{x^2-x}} + \int_2^{+\infty} \frac{dx}{\sqrt{x^2-x}}$$

A antiderivada de f será:

$$\int \frac{dx}{\sqrt{x^2-x}} = ?$$

$$x^2 - x = (x+k)^2 + l$$
$$= x^2 + 2kx + k^2 + l$$

$$\begin{cases} 2k = -1 \Rightarrow k = -\frac{1}{2} \\ k^2 + l = 0 \Rightarrow l = -\frac{1}{4} \end{cases}$$

$$\Rightarrow x^2 - x = \left(x - \frac{1}{2}\right)^2 - \frac{1}{4}$$

Disso, vem que:

$$\int \frac{dx}{\sqrt{x^2-x}} = \int \frac{dx}{\sqrt{\left(x-\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} =$$

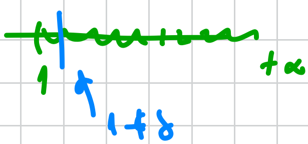
$$\int \frac{du}{\sqrt{u^2-a^2}} = \ln\left|u+\sqrt{u^2-a^2}\right| + c$$

$$= \ln\left|\left(x-\frac{1}{2}\right)+\sqrt{\left(x-\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}\right| + c = \ln\left|x-\frac{1}{2}+\sqrt{x^2-x}\right| + c$$

Assim:

$$\int_1^{+\infty} f(x)\,dx = \int_1^2 f(x)\,dx + \int_2^{+\infty} f(x)\,dx =$$

$$= \lim_{\delta \to 0^+} \int_{1+\delta}^2 f(x)\,dx + \lim_{b \to +\infty} \int_2^b f(x)\,dx =$$



$$= \lim_{\delta \to 0^+} \ln\left|x-\frac{1}{2}+\sqrt{x^2-x}\right| \Bigg|_{1+\delta}^{2} +$$

$$+\lim_{b\to+\infty} \ln\left|x-\frac{1}{2}+\sqrt{x^2-x}\right|\Bigg|_2^b =$$

$$\lim_{\delta\to0^+} \cancel{\ln\left|2-\frac{1}{2}+\sqrt{4-2}\right|} - \ln\left|1+\delta-\frac{1}{2}+\sqrt{(1+\delta)^2-1-\delta}\right| +$$

$$+\lim_{b\to+\infty} \ln\left|b-\frac{1}{2}+\sqrt{b^2-b}\right| - \cancel{\ln\left|2-\frac{1}{2}+\sqrt{4-2}\right|}$$

$$=\lim_{\delta\to0^+} -\ln\left|\frac{1}{2}+\delta+\sqrt{\cancel{1}+2\delta+\delta^2-\cancel{1}-\delta}\right| +$$

$$+\lim_{b\to+\infty}\left|b-\frac{1}{2}+\sqrt{b^2-b}\right|$$

$$=\lim_{\delta\to0^+} -\ln\left|\frac{1}{2}+\underbrace{\delta}_{\to0}+\underbrace{\sqrt{\delta^2+\delta}}_{\to0}\right| + \lim_{b\to+\infty}\left|b-\frac{1}{2}+\sqrt{b^2-b}\right|$$

$$=-\ln\frac{1}{2}+\lim_{b\to+\infty}\ln\left|\underbrace{\underbrace{b-\frac{1}{2}}_{+\infty}+\underbrace{\sqrt{b(b-1)}}_{+\infty}}_{+\infty}\right| = +\infty$$

Ou seja, esta integral diverge.

De uma prova de 2024:

Questão 01. Calcule a seguinte integral imprópria, se existir: $\displaystyle\int_0^{+\infty} xe^{-x^2}\,dx$.

$$f(x) = x \cdot e^{-x^2} = \frac{x}{e^{x^2}}. \quad D(f) = \mathbb{R}.$$

$$\int_0^{+\infty} x \cdot e^{-x^2}\,dx = \lim_{b \to +\infty} \int_0^b x \cdot e^{-x^2}\,dx.$$

$$\int x\,e^{-x^2}\,dx = \int e^{v}\,dv$$

$$v = -x^2 \Rightarrow dv = -2x\,dx$$

$$= -\frac{1}{2}\int e^{-x^2}(-2x\,dx) = \underbrace{-\frac{1}{2} \cdot e^{-x^2} + C}$$

Disso, temos:

$$\underbrace{\int_0^{+\infty} x \cdot e^{-x^2}\,dx} = \lim_{b \to +\infty} \int_0^b x \cdot e^{-x^2}\,dx = \lim_{b \to +\infty} \left(-\frac{1}{2}\,e^{-x^2}\right)\Big|_0^b =$$

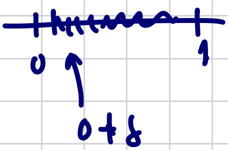
$$= \lim_{b \to +\infty} -\frac{1}{2}\,e^{-b^2} - \left(-\frac{1}{2}\,e^{0}\right) =$$

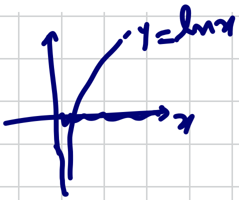
$$= \lim_{b \to +\infty} - \underbrace{\frac{1}{2e^{b^2}}}_{0} + \frac{1}{2} = \frac{1}{2}$$

$$\Rightarrow \int_0^{+\infty} x e^{-x^2} dx = \frac{1}{2}$$

Lista 6!

7. Calcule a integral $\int_0^1 \ln x \, dx$, se esta integral existir.





$$\int_0^1 \ln x \, dx = \lim_{\delta \to 0^+} \int_\delta^1 \ln x \, dx \quad ; \quad \text{onde:}$$

$$\int \ln x \, dx = \int u \, dv = u \cdot v - \int v \, du$$

$$\begin{cases} u = \ln x \Rightarrow du = \dfrac{dx}{x} \\ dv = dx \Rightarrow v = x \end{cases}$$

$$\Rightarrow \int \ln x\,dx = x\cdot\ln x - \int \cancel{x}\cdot\frac{dx}{\cancel{x}} = x\ln x - x + C.$$

Assim, teremos:

$$\int_0^1 \ln x\,dx = \lim_{\delta\to 0^+}\int_\delta^1 \ln x\,dx = \lim_{\delta\to 0^+}(x\ln x - x)\Big|_\delta^1 =$$

$$= \lim_{\delta\to 0^+}(\underbrace{1\ln 1}_{=0} - 1) - (\delta\ln\delta - \delta) =$$

$$= -1 - \lim_{\delta\to 0^+}(\underbrace{\delta\ln\delta}_{\substack{0\cdot\infty \\ \text{INDET.}}} - \delta) =$$

$$= -1 - \lim_{\delta\to 0^+}\frac{\ln\delta}{\frac{1}{\delta}} - \lim_{\delta\to 0^+}\delta$$

$$= -1 - \lim_{\delta\to 0^+}\frac{\ln\delta}{\delta^{-1}} - 0 =$$

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$$= 1 - \lim_{\delta\to 0^+}\frac{\frac{1}{\cancel{\delta}}}{-\frac{1}{\delta^{\cancel{2}}}} = 1 - \lim_{\delta\to 0^+}\frac{1}{\cancel{\delta}}\cdot\left(-\frac{\delta^{\cancel{2}}}{1}\right) = 1$$