CÁLCULO II

11/07/25 - AULA 18

INTEGRAIS DE FUNÇÕES DA FORMA $\int R(x^{\frac{1}{n}})\,dx$.

Funções da forma $R(x^{\frac{1}{n}})$ são as funções irracionais fracionárias. Por exemplo:

$$\int \frac{\sqrt{x} - \sqrt[3]{x}}{\sqrt[4]{x} - \sqrt{x}}\,dx = \int \frac{(x^{\frac{1}{2}} - x^{\frac{1}{3}})\,dx}{x^{\frac{1}{4}} - x^{\frac{1}{2}}}.$$

ou $\displaystyle\int \frac{\sqrt{x-1} - 1}{\sqrt{x-1} + 1}\,dx.$

Como resolver integrais desse tipo?

Nestes casos, faz-se uma mudança de variável. Em geral, faz-se $t = x^n$, onde n = mmc dos índices dos radicais.

EX.: $\displaystyle\int \frac{\sqrt{x} - \sqrt[3]{x}}{\sqrt[4]{x} - \sqrt{x}}\,dx = ?$

Neste caso, faz-se $mmc(2,3,4) = ?$

$$\begin{array}{ccc|c} 2 & 3 & 4 & 2 \\ 1 & 3 & 2 & 2 \\ 1 & 3 & 1 & 3 \\ 1 & 1 & 1 & \\ \hline & & & 12 \end{array}$$

Então, escreva $\boxed{x = p^{12}} \Rightarrow dx = 12p^{11}.\,dp$

Assim:

$$\int \frac{\sqrt{x} - \sqrt[3]{x}}{\sqrt[4]{x} - \sqrt{x}}\,dx = \int \frac{\sqrt{p^{12}} - \sqrt[3]{p^{12}}}{\sqrt[4]{p^{12}} - \sqrt{p^{12}}} \cdot 12p^{11}.\,dp =$$

$$= \int \frac{p^6 - p^4}{p^3 - p^6} \cdot 12p^{11}.\,dp = \int \frac{p^{\cancel{4}}(p^2-1)}{\cancel{p^3}(1-p^3)} \cdot 12p^{11}.\,dp$$

$$= -\int \frac{12p^{12}.(p^2-1)}{p^3-1}\,dp =$$

$$= -12\int \frac{p^{12}.(p+1)\cancel{(p-1)}}{\cancel{(p-1)}(p^2+p+1)}.\,dp = -12\int \frac{p^{12}(p+1)\,dp}{\underbrace{p^2+p+1}_{\text{NÃO POSSUI RAÍZES REAIS}}} =$$

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}.b + a^{n-3}.b^2 + \dots + b^{n-1})$$

$$a^2 - b^2 = (a-b).(a+b)$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

$$= -12 \int \frac{p^{13}+p^{12}}{p^2+p+1}\, dp$$

$$
\begin{array}{l|l}
p^{13} + p^{12} & p^2+p+1 \\
\hline
-\cancel{p^{13}} - \cancel{p^{12}} - p^{11} & \underbrace{p^{11} - p^9 + p^8 - p^6 + p^5 - p^3 + p^2 - 1}_{a(x)} \\
-\cancel{p^{11}} & \\
+\cancel{p^{11}} + p^{10} + p^9 & \\
\cancel{p^{10}} + \cancel{p^9} & \\
-\cancel{p^{10}} - \cancel{p^9} - p^8 & \\
-\cancel{p^8} & \\
+\cancel{p^8} + p^7 + p^6 & \\
\cancel{p^7} + \cancel{p^6} & \\
-\cancel{p^7} - \cancel{p^6} - p^5 & \\
-\cancel{p^5} & \\
+\cancel{p^5} + p^4 + p^3 & \\
\cancel{p^4} + \cancel{p^3} & \\
-p^4 - p^3 - p^2 & \\
-\cancel{p^2} & \\
+\cancel{p^2} + p + 1 & \\
\hline
p+1 &
\end{array}
$$

$$\int \frac{\text{divid.}}{\text{divisor}} = \int \frac{q \cdot \text{divisor} + R}{\text{divisor}} =$$

$$\begin{array}{l|l} \text{dividendo} & \text{divisor} \\ \hline R & q \end{array}$$

$$= \int q + \frac{R}{\text{divisor}}$$

Então:

$$-12\int \frac{(p^{13}+p^{12})\,dp}{p^2+p+1} = -12\cdot\int\left(p^{11}-p^9+p^8-p^6+p^5-p^3+p^2-1+\frac{p+1}{p^2+p+1}\right)dp$$

$$= -12\int p^{11}\,dp + 12\int p^9\,dp - 12\int p^8\,dp + 12\int p^6\,dp - 12\int p^5\,dp +$$

$$+ 12\int p^3\,dp - 12\int p^2\,dp + 12\int dp - 12\int \frac{p+1}{p^2+p+1}\,dp$$

$$= -\cancel{12}\frac{p^{12}}{\cancel{12}} + \overset{6}{\cancel{12}}\frac{p^{10}}{\underset{5}{\cancel{10}}} - \overset{4}{\cancel{12}}\frac{p^9}{\underset{3}{\cancel{9}}} + 12\frac{p^7}{7} - \overset{2}{\cancel{12}}\frac{p^6}{\underset{1}{\cancel{6}}} + \overset{3}{\cancel{12}}\frac{p^4}{\cancel{4}} - \overset{4}{\cancel{12}}\frac{p^3}{\underset{1}{\cancel{3}}} +$$

$$+ 12p - 12\int \frac{p+1}{p^2+p+1}\,dp =$$

$$= -p^{12} + \frac{6}{5}p^{10} - \frac{4}{3}p^9 + \frac{12}{7}p^7 - 2p^6 + 3p^4 - 4p^3 + 12p - 12\underbrace{\int \frac{(p+1)\,dp}{p^2+p+1}}_{(*)}$$

$$(*) \quad \int \frac{p+1}{p^2+p+1}\cdot dp = \int \frac{\frac{1}{2}(2p+1)+\frac{1}{2}}{p^2+p+1}\cdot dp =$$

$$r = p^2+p+1 \Rightarrow dr = (2p+1)\,dp$$

$$= \frac{1}{2}\int \frac{(2p+1)\,dp}{p^2+p+1} + \frac{1}{2}\int \frac{dp}{p^2+p+1}$$

$$= \frac{1}{2}\ln|p^2+p+1| + \frac{1}{2}\int \frac{dp}{p^2+p+1} =$$

$$p^2+p+1 = (p+k)^2+l$$

$$p^2+2kp+k^2+l$$

$$\begin{cases} 2k = 1 \Rightarrow k = \frac{1}{2} \\ k^2+l = 1 \end{cases}$$

$$\frac{1}{4}+l = 1 \Rightarrow l = \frac{3}{4}$$

$$\Rightarrow p^2+p+1 = \left(p+\frac{1}{2}\right)^2+\frac{3}{4}$$

$$= \frac{1}{2}\ln(\underbrace{p^2+p+1}_{>0}) + \frac{1}{2}\int \frac{dp}{\left(p+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} =$$

$p^2+p+1=0$
não tem
raízes
reais

$$\int \frac{dw}{w^2+a^2} = \frac{1}{a}\cdot \arctan\left(\frac{w}{a}\right)+c$$

$$= \frac{1}{2}\ln(p^2+p+1) + \frac{1}{\cancel{2}}\,\frac{1}{\frac{\sqrt{3}}{\cancel{2}}}\cdot\arctan\left(\frac{p+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)+c$$

$$= \frac{1}{2}\ln(p^2+p+1) + \frac{1}{\sqrt{3}}\cdot\arctan\left(\frac{2p+1}{\sqrt{3}}\right)+c$$

Di ciò, segue:

$$\int \frac{\sqrt{x}-\sqrt[3]{x}}{\sqrt[4]{x}-\sqrt{x}}\,dx = -12\int \frac{p^{13}+p^{12}}{p^2+p+1}\,dp =$$

$$= -p^{12}+\frac{6}{5}p^{10}-\frac{4}{3}p^{9}+\frac{12}{7}p^{7}-2p^{6}+3p^{4}-4p^{3}+12p-12\underbrace{\int\frac{(p+1)\,dp}{p^2+p+1}}_{(*)}$$

$$= -p^{12}+\frac{6}{5}p^{10}-\frac{4}{3}p^{9}+\frac{12}{7}p^{7}-2p^{6}+3p^{4}-4p^{3}+12p-$$

$$-12\cdot\left(\frac{1}{2}\ln(p^2+p+1)+\frac{1}{\sqrt{3}}\cdot\arctan\left(\frac{2p+1}{\sqrt{3}}\right)+c\right)$$

$$= -p^{12}+\frac{6}{5}p^{10}-\frac{4}{3}p^{9}+\frac{12}{7}p^{7}-2p^{6}+3p^{4}-4p^{3}+12p-$$

$$-6\ln(p^2+p+1)-\frac{12}{\sqrt{3}}\cdot\arctan\left(\frac{2p+1}{\sqrt{3}}\right)+c =$$

$$= -\left(\sqrt[\cancel{12}]{x}\right)^{\cancel{12}}+\frac{6}{5}\left(\sqrt[\cancel{12}^{6}]{x}\right)^{\cancel{10}^{5}}-\frac{4}{3}\left(\sqrt[\cancel{12}^{4}]{x}\right)^{\cancel{9}^{3}}+\frac{12}{7}\left(\sqrt[12]{x}\right)^{7}-2\left(\sqrt[\cancel{12}^{2}]{x}\right)^{\cancel{6}}+$$

$$+3\left(\sqrt[\cancel{12}^{3}]{x}\right)^{\cancel{4}}-4\left(\sqrt[\cancel{12}^{4}]{x}\right)^{\cancel{3}}+12\sqrt[12]{x}-$$

$$-6\ln\left(\left(\sqrt[\cancel{12}^{6}]{x}\right)^{\cancel{2}}+\sqrt[12]{x}+1\right)-\frac{12}{\sqrt{3}}\arctan\left(\frac{2\sqrt[12]{x}+1}{\sqrt{3}}\right)$$

$$x=p^{12}$$
$$\Downarrow$$
$$p=\sqrt[12]{x}$$

$$= -x + \frac{6}{5}\sqrt[6]{x^5} - \frac{4}{3}\sqrt[4]{x^3} + \frac{12}{7}\sqrt[12]{x^7} - 2\sqrt{x} + 3\sqrt[3]{x} -$$

$$- 4\sqrt[4]{x} + 12\sqrt[12]{x} - 6\ln\left(\sqrt[6]{x} + \sqrt[12]{x} + 1\right) -$$

$$- \frac{12}{\sqrt{3}} \arctan\left(\frac{2\sqrt[12]{x} + 1}{\sqrt{3}}\right) + c.$$

02) $\displaystyle\int \frac{\sqrt{x-1} - 1}{\sqrt{x-1} + 1}\, dx = ?$

Solução: Escreva $\boxed{x - 1 = p^2} \Rightarrow x = p^2 + 1$

$\Rightarrow dx = 2p \cdot dp$

Assim, teremos:

$$\int \frac{\sqrt{x-1} - 1}{\sqrt{x-1} + 1}\, dx = \int \frac{\sqrt{p^2} - 1}{\sqrt{p^2} + 1} \cdot 2p \cdot dp =$$

$$= \int \frac{(p-1)2p\,dp}{p+1} = \int \frac{(2p^2 - 2p)\,dp}{p+1}$$

$$\begin{array}{r|l} 2p^2 - 2p & p+1 \\ \cline{2-2} -2p^2 - 2p & 2p - 4 \\ \hline -4p & \\ +4p + 4 & \\ \hline 4 & \end{array}$$

$$\Rightarrow 2p^2 - 2p = (p+1)\cdot(2p-4) + 4$$

Dimo:

$$\int \frac{2p^2-2p}{p+1}\,dp = \int \left(\frac{\cancel{(p+1)}(2p-4)+4}{\cancel{p+1}} \right) dp =$$

$$= \int \left(2p-4 + \frac{4}{p+1} \right) dp =$$

$$= 2\int p\,dp - 4\int dp + 4 \cdot \int \frac{dp}{p+1} =$$

$$= \frac{\cancel{2}p^2}{\cancel{2}} - 4p + 4 \cdot \ln|p+1| + C =$$

$$= p^2 - 4p + 4\ln|p+1| + C =$$

$x-1=p^2$

$\Rightarrow p = \sqrt{x-1}$

$$= \left(\sqrt{x-1}\right)^{\cancel{2}} - 4\sqrt{x-1} + 4 \cdot \ln\left|\sqrt{x-1}+1\right| + C$$

$$= x - 1 - 4\sqrt{x-1} + 4\ln\left|\sqrt{x-1}+1\right| + C$$

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$$= x - 4\sqrt{x-1} + 4 \cdot \ln\left|\sqrt{x-1}+1\right| + C.$$

03) $\displaystyle\int \frac{\sqrt{x}\,dx}{\sqrt[4]{x^3}+1} = ?$ mmc $(2,4) = 4$

Escreva $\boxed{x = p^4}$. Assim $dx = 4p^3\,dp$ e disso:

$$\int \frac{\sqrt{x}\,dx}{\sqrt[4]{x^3}+1} = \int \frac{\sqrt{p^{\cancel{4}\,2}}\cdot 4p^3\,dp}{\sqrt[\cancel{4}]{(p^{\cancel{4}})^3}+1} =$$

$$\int \frac{4\cdot p^2\cdot p^3\cdot dp}{p^3+1} = 4\int \frac{p^5\,dp}{p^3+1} =$$

$$\begin{array}{r|l} p^{\cancel{5}} & p^3+1 \\ \hline -p^{\cancel{5}} - p^2 & p^2 \\ \hline -p^2 & \end{array}$$

$\Rightarrow p^5 = (p^3+1)\cdot p^2 - p^2$

$$4\cdot\int \frac{(p^3+1)p^2 - p^2}{p^3+1}\,dp = 4\int\left(p^2 - \frac{p^2}{p^3+1}\right)dp =$$

$$4\int p^2\,dp - \frac{4}{3}\cdot\int \frac{3\cdot p^2\,dp}{p^3+1} =$$

$\displaystyle\int \frac{dw}{w}$

$w = p^3+1$

$\hookrightarrow dw = 3p^2\,dp$

$$= 4\frac{p^3}{3} - \frac{4}{3} \cdot \ln|p^3+1| + C =$$

$$= 4 \cdot \frac{(\sqrt[4]{x})^3}{3} - \frac{4}{3} \ln|(\sqrt[4]{x})^3 + 1| + C =$$

$x = p^4$
$\hookrightarrow p = \sqrt[4]{x}$

$$= \frac{4\sqrt[4]{x^3}}{3} - \frac{4}{3} \cdot \ln|\sqrt[4]{x^3} + 1| + C$$

d) $\int \sqrt{\frac{1-x}{1+x}}\, dx = ?$

Escreva $\boxed{p^2 = \frac{1-x}{1+x}}$. Então:

$p = \sqrt{\frac{1-x}{1+x}}$

$p^2(1+x) = 1-x$

$p^2 + x \cdot p^2 = 1-x$

$x \cdot p^2 + x = 1 - p^2$

$x(p^2+1) = 1-p^2$

$$\Rightarrow x = \frac{1-p^2}{1+p^2} \Rightarrow dx = \frac{(1+p^2)(-2p) - (1-p^2)2p}{(1+p^2)^2} \cdot dp$$

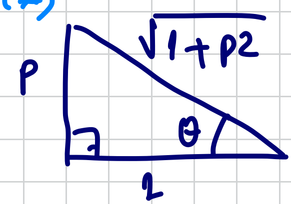
$$\Rightarrow dx = \frac{-2p - \cancel{2p^3} - 2p + \cancel{2p^3}}{(1+p^2)^2}\,dp$$

$$\Rightarrow dx = \frac{-4p}{(1+p^2)^2}\,dp$$

Disso, obtemos:

$$\int \sqrt{\frac{1-x}{1+x}}\,dx = \int \cancel{\sqrt{\cancel{p^2}}} \cdot \frac{(-4p)}{(1+p^2)^2}\,dp =$$

$$= \underbrace{4\int \frac{p^2}{(1+p^2)^2}\,dp}_{(*)} \quad ,$$



$$\tan\theta = \frac{p}{1}$$

$$p = \tan\theta.$$

$$\Rightarrow dp = \sec^2\theta\, d\theta.$$

$$\cos\theta = \frac{1}{\sqrt{1+p^2}}$$

$$\sqrt{1+p^2} = \sec\theta.$$

$$1+p^2 = \sec^2\theta.$$

$$\operatorname{sen}\theta = \frac{p}{\sqrt{1+p^2}}$$

Disso, obtemos:

(*) $$\int \frac{p^2}{(1+p^2)^2}\,dp = \int \frac{\tan^2\theta}{(\sec^2\theta)^{\cancel{2}}} \cdot \cancel{\sec^2\theta}\,d\theta =$$

$$= \int \tan^2\theta \cdot \cos^2\theta \, d\theta = \int \frac{\operatorname{sen}^2\theta}{\cancel{\cos^2\theta}} \cdot \cancel{\cos^2\theta}\,d\theta =$$

$$= \int \operatorname{sen}^2\theta \, d\theta = \int \frac{1-\cos 2\theta}{2}\,d\theta =$$

$$\operatorname{sen}^2\theta = \frac{1-\cos 2\theta}{2}$$

$$= \frac{1}{2}\int d\theta - \frac{1}{2}\cdot\frac{1}{2}\int \cos 2\theta \,(2\,d\theta) =$$

$$= \frac{1}{2}\theta - \frac{1}{4}\cdot \operatorname{sen} 2\theta + C = \frac{1}{2}\theta + \frac{1}{4}\cdot 2\operatorname{sen}\theta \cos\theta + C$$

$$\operatorname{sen} 2\theta = 2 \cdot \operatorname{sen}\theta \cdot \cos\theta$$

$$= \frac{1}{2}\theta + \frac{1}{2}\cdot \operatorname{sen}\theta \cdot \cos\theta + C =$$

$$= \frac{1}{2}\cdot \arctan(p) + \frac{1}{2}\cdot \frac{p}{\sqrt{1+p^2}} \cdot \frac{1}{\sqrt{1+p^2}} + C$$

$$= \frac{1}{2}\arctan(p) + \frac{p}{2(1+p^2)} + C$$

Disso, obtemos:

$$\int \sqrt{\frac{1-x}{1+x}}\,dx = 4\cdot \int \frac{p^2}{(1+p^2)^2}\,dp = 4\cdot\left(\frac{1}{2}\arctan p + \frac{p}{2(1+p^2)}\right) + C$$

$$= 2\arctan p + \frac{2p}{1+p^2} + C =$$

$$p = \sqrt{\frac{1-x}{1+x}}$$

$$= 2\arctan\sqrt{\frac{1-x}{1+x}} + \frac{2\sqrt{\frac{1-x}{1+x}}}{1+\left(\sqrt{\frac{1-x}{1+x}}\right)^2} + C$$

$$= 2\arctan\sqrt{\frac{1-x}{1+x}} + \frac{2\sqrt{\frac{1-x}{1+x}}}{1+\frac{1-x}{1+x}} + C$$

$$= 2\arctan\sqrt{\frac{1-x}{1+x}} + \frac{2\sqrt{\frac{1-x}{1+x}}}{\frac{1+\cancel{x}+1-\cancel{x}}{1+x}} + C$$

$$= 2\arctan\sqrt{\frac{1-x}{1+x}} + \frac{1}{1+x}\sqrt{\frac{1-x}{1+x}} + C$$