CÁLCULO II

09/07/25 - AULA 17

LISTA 05:

3. Mostre que:

(a) $\displaystyle\int_2^4 \frac{x^3-2}{x^3-x^2}dx = \frac{5}{2} + \ln\frac{4}{3}$

(b) $\displaystyle\int_0^5 \frac{(x^2-3)dx}{(x+2)(x+1)^2} = \ln\frac{7}{2} - \frac{5}{3}$

(a) $\displaystyle\int_2^4 \frac{x^3-2}{x^3-x^2}\,dx$

$$x^3 - x^2 = x^2(x-1)$$

Obs.: Quando temos $\displaystyle\int \frac{P(x)}{q(x)}dx$ e P e q são polinômios de mesmo grau, primeiro deve-se efetuar uma divisão (se o grau do numerador também fosse MAIOR do que o grau do denominador, também faríamos uma divisão). Assim:

$$\begin{array}{l|l} P(x) & q(x) \\ \hline R(x) & a(x) \end{array}$$

$\Rightarrow P(x) = q(x)\cdot a(x) + R(x)$

$\Rightarrow \dfrac{P(x)}{q(x)} = \dfrac{\cancel{q(x)}\cdot a(x)}{\cancel{q(x)}} + \dfrac{R(x)}{q(x)}$

$$\frac{P(x)}{q(x)} = a(x) + \frac{R(x)}{q(x)}$$

$$\int \frac{P(x)}{q(x)} = \int a(x) + \int \frac{R(x)}{q(x)}$$

Voltando ao problema:

1º: $\int \frac{x^3-2}{x^3-x^2}\,dx$

$$\begin{array}{r|l} x^3 - 2 & x^3 - x^2 \\ \cline{2-2} -x^3 + x^2 & 1 \\ \hline x^2 - 2 & \end{array}$$

$$\Rightarrow x^3-2 = 1\cdot(x^3-x^2) + x^2-2$$

$$\Rightarrow \frac{x^3-2}{x^3-x^2} = \frac{1\cdot\cancel{(x^3-x^2)}}{\cancel{x^3-x^2}} + \frac{x^2-2}{x^3-x^2}$$

$$= 1 + \frac{x^2-2}{x^3-x^2}$$

Arsim:

$$\int \frac{x^3-2}{x^3-x^2}\,dx = \int \left(1 + \frac{x^2-2}{x^3-x^2}\right)dx =$$

$$= \int dx + \int \frac{x^2-2}{x^3-x^2}\,dx =$$

$$= x + \int \frac{x^2-2}{x^2(x-1)}\,dx$$

$$\frac{x^2-2}{x^2\cancel{(x-1)}} = \frac{A}{x^2} + \frac{B}{x} + \frac{C}{x-1}$$

$$= \frac{A(x-1) + Bx(x-1) + Cx^2}{x^2\cancel{(x-1)}}$$

$$1x^2-2 = Ax - A + Bx^2 - Bx + Cx^2$$

$$\begin{cases} B + C = 1 \\ A - B = 0 \\ -A = -2 \Rightarrow A = 2 \end{cases}$$

$$2 - B = 0 \Rightarrow B = 2$$

$$B + C = 1$$

$$2 + C = 1 \implies C = -1$$

$$\Rightarrow \int \frac{x^2-2}{x^2(x-1)}\,dx = \int \frac{2}{x^2}\,dx + \int \frac{2}{x}\,dx + \int \frac{-1}{x-1}\,dx$$

$$= 2\int x^{-2}dx + 2\int \frac{dx}{x} - \int \frac{dx}{x-1}$$

$$= 2\,\frac{x^{-1}}{-1} + 2\ln|x| - \ln|x-1| + C$$

$$= \underbrace{-\frac{2}{x} + 2\ln|x| - \ln|x-1| + C}$$

Então:

$$\int \frac{x^3-2}{x^3-x^3} = x + \underbrace{\int \frac{x^2-2}{x^3-x^2}\,dx} =$$

$$= x - \frac{2}{x} + 2\ln|x| - \ln|x-1| + C$$

Por fim, com o T.F.C., tem-se:

$$\int_2^4 \frac{x^3-2}{x^3-x^2}\,dx = \left(x - \frac{2}{x} + 2\ln|x| - \ln|x-1| \right)\Bigg|_2^4 =$$

$$= 4 - \frac{2}{4} + 2\ln|4| - \ln|3| -$$

$$- \left(2 - \frac{2}{2} + 2\ln|2| - \ln|2-1| \right)$$

$$= 4 - \frac{1}{2} + 2\ln 4 - \ln 3 - 2 + 1 - 2\ln 2 - \underbrace{\ln 1}_{=0}$$

$$= \frac{5}{2} + 2\ln 4 - \ln 3 - 2\ln 2$$

$$= \frac{5}{2} + 2\ln 2^2 - \ln 3 - 2\ln 2$$

$$= \frac{5}{2} + 4\ln 2 - 2\ln 2 - \ln 3$$

$$= \frac{5}{2} + 2\ln 2 - \ln 3 = \frac{5}{2} + \ln 4 - \ln 3$$

$$= \frac{5}{2} + \ln\frac{4}{3}$$

INTEGRAIS TRIGONOMÉTRICAS VIA TROCA UNIVERSAL

Dado $\int R(\operatorname{sen} x, \cos x)\,dx$, ou seja, de uma função racional em termos de sen x e cos x, é possível resolvê-la fazendo uma substituição especial de variável.

Escreva $\boxed{z = \tan \frac{x}{2}}$. Então;

Diferenciando, vem:

$$dz = \sec^2 \frac{x}{2} \cdot \frac{1}{2}\,dx$$

Além disso, da relação trigonométrica:

$$1 + \tan^2 \alpha = \sec^2 \alpha \,; \quad \text{temos:}$$

$$1 + \underbrace{\tan^2 \frac{x}{2}}_{z^2} = \sec^2 \frac{x}{2}$$

$$1 + z^2 = \sec^2 \frac{x}{2} \quad \Rightarrow$$

Disso;

$$dz = \underbrace{\sec^2\frac{x}{2}}_{1+z^2} \cdot dx$$

$$\Rightarrow dz = (1+z^2) \cdot dx \Rightarrow \boxed{dx = \frac{dz}{1+z^2}}$$

Além disso:

$$\operatorname{sen} x = 2 \cdot \operatorname{sen}\frac{x}{2} \cdot \cos\frac{x}{2}$$

$$[\operatorname{sen} 2\alpha = 2 \operatorname{sen}\alpha \cdot \cos\alpha] \quad ;$$

e como $\sec^2\frac{x}{2} = 1+z^2 \Rightarrow \cos^2\frac{x}{2} = \frac{1}{1+z^2}$

$$\Rightarrow \cos\frac{x}{2} = \sqrt{\frac{1}{1+z^2}} = \frac{1}{\sqrt{1+z^2}}$$

$$\operatorname{sen}\frac{x}{2} = \sqrt{1-\cos^2\frac{x}{2}} = \sqrt{1-\left(\frac{1}{\sqrt{1+z^2}}\right)^2}$$

$$= \sqrt{1-\frac{1}{1+z^2}} = \sqrt{\frac{\cancel{1}+z^2-\cancel{1}}{1+z^2}}$$

$$= \frac{z}{\sqrt{1+z^2}}$$

Assim:

$$\operatorname{sen} x = 2 \cdot \operatorname{sen} \frac{x}{2} \cdot \cos \frac{x}{2} = 2 \cdot \frac{z}{\sqrt{1+z^2}} \cdot \frac{1}{\sqrt{1+z^2}}$$

$$\Rightarrow \boxed{\operatorname{sen} x = \frac{2z}{1+z^2}}$$

Disso:

$$\cos x = \sqrt{1 - \operatorname{sen}^2 x} = \sqrt{1 - \left(\frac{2z}{1+z^2}\right)^2}$$

$$= \sqrt{1 - \frac{4z^2}{(1+z^2)^2}} = \sqrt{\frac{(1+z^2)^2 - 4z^2}{(1+z^2)^2}}$$

$$= \frac{\sqrt{1 + 2z^2 + z^4 - 4z^2}}{1+z^2} = \frac{\sqrt{1 - 2z^2 + z^4}}{1+z^2} =$$

$$= \frac{\sqrt{(1-z^2)^2}}{1+z^2} = \frac{1-z^2}{1+z^2}$$

$$\Rightarrow \boxed{\cos x = \frac{1-z^2}{1+z^2}}$$

<u>CONCLUSÃO</u>: escrevendo

$$\boxed{z = \tan \frac{x}{2}},$$

obtemos:

$$\boxed{dx = \frac{dz}{1+z^2}} \; ; \quad \boxed{\operatorname{sen} x = \frac{2z}{1+z^2}} \quad e \quad \boxed{\cos x = \frac{1-z^2}{1+z^2}}$$

Vejamos alguns exemplos:

01) Calcule $\displaystyle\int \frac{dx}{\operatorname{sen} x - \cos x}$.

<u>Solução:</u>

$$\int \frac{dx}{\operatorname{sen} x - \cos x} = \int \frac{\frac{dz}{1+z^2}}{\frac{2z}{1+z^2} - \frac{1-z^2}{1+z^2}} =$$

$$= \int \frac{\frac{dz}{\cancel{1+z^2}}}{\frac{2z-1+z^2}{\cancel{1+z^2}}} = \int \frac{dz}{z^2+2z-1} =$$

$$z^2+2z-1 = (z+k)^2+l$$
$$= z^2+2kz+k^2+l$$

$$\Leftrightarrow \begin{cases} 2k=2 \Rightarrow k=1 \\ k^2+l=-1 \end{cases}$$

$$(1)^2+l=-1$$
$$l=-2$$

$$\Rightarrow z^2+2z-1=(z+1)^2-2$$

$$\overset{(3)}{=} \int \frac{dz}{(z+1)^2-2} \quad \frac{1}{2\sqrt{2}} \cdot \ln\left|\frac{z+1-\sqrt{2}}{z+1+\sqrt{2}}\right| + c =$$

$$\int \frac{dr}{r^2-a^2} = \frac{1}{2a} \ln\left|\frac{r-a}{r+a}\right| + c \qquad a=\sqrt{2}$$

$$r = z+1 \Rightarrow dr = dz$$

$$= \frac{1}{2\sqrt{2}} \cdot \ln\left|\frac{\tan\frac{x}{2}+1-\sqrt{2}}{\tan\frac{x}{2}+1+\sqrt{2}}\right| + c$$

$$z = \tan\frac{x}{2}$$

02) $\displaystyle\int \frac{\operatorname{sen} x \, dx}{1+\operatorname{sen} x} = ?$

Solução: $\displaystyle\int \frac{\operatorname{sen} x \, dx}{1+\operatorname{sen} x} = \int \frac{\frac{2z}{1+z^2} \cdot \frac{dz}{1+z^2}}{1 + \frac{2z}{1+z^2}} =$

$$= \int \frac{\frac{2z\,dz}{(1+z^2)^2}}{\frac{1+z^2+2z}{1+z^2}} = \int \frac{2z}{(1+z^2)^{\cancel{2}}} \cdot \frac{\cancel{1+z^2}}{(z+1)^2} \, dz =$$

$$\int \frac{2z\,dz}{(1+z^2)(z+1)^2}$$

$$\frac{2z}{(1+z^2)(z+1)^2} = \frac{Az+B}{1+z^2} + \frac{C}{(z+1)^2} + \frac{D}{z+1}$$

$$= \frac{(Az+B)(1+z)^2 + C(1+z^2) + D(z+1)(1+z^2)}{(1+z^2)\cdot(z+1)^2}$$

$$2z = (Az+B)(1+2z+z^2) + C + Cz^2 + Dz + Dz^3 + D + Dz^2$$

$$2z = Az + 2Az^2 + Az^3 + B + 2Bz + Bz^2 + C + Cz^2 + Dz + Dz^3 + D + Dz^2$$

$$\begin{cases} A + D = 0 \\ 2A + \boxed{B + C + D} = 0 \\ A + 2B + D = 2 \\ \boxed{B + C + D = 0} \end{cases}$$

$$2A + 0 = 0 \Rightarrow A = 0$$

$$A + D = 0 \Rightarrow \boxed{D = 0}$$

$$A = 0$$

$$A + 2B + D = 2$$
$$0 + 2B + 0 = 2 \Rightarrow B = 1$$

$$B + C + D = 0 \Rightarrow 1 + C + 0 = 0$$
$$\Rightarrow \boxed{C = -1}$$

Assim, obtemos:

$$\frac{2z}{(1+z^2)(z+1)^2} = \frac{0z+1}{1+z^2} + \frac{-1}{(z+1)^2} + \frac{0}{z+1}$$

$$\Rightarrow \int \frac{2z\,dz}{(1+z^2)(z+1)} = \int \frac{1}{1+z^2}\,dz - \int \frac{dz}{(z+1)^2} =$$

$$= \int \frac{dz}{z^2+1} - \int (z+1)^{-2}\, dz$$

$$= \frac{1}{1} \cdot \arctan\left(\frac{z}{1}\right) - \frac{(z+1)^{-1}}{-1} + C$$

$$= \arctan z + \frac{1}{z+1} + C =$$

$z = \tan\frac{x}{2}$

$$= \arctan\left(\tan\frac{x}{2}\right) + \frac{1}{\tan\frac{x}{2}+1} + C$$

$$= \frac{x}{2} + \frac{1}{\tan\frac{x}{2}+1} + C$$

03) $\displaystyle\int \frac{2-\operatorname{sen}x}{2+\cos x}\,dx = ?$

$$\int \frac{2-\operatorname{sen}x}{2+\cos x} = \int \frac{2-\frac{2z}{1+z^2}}{2+\frac{1-z^2}{1+z^2}} \cdot \frac{dz}{1+z^2} =$$

$$= \int \frac{\frac{2+2z^2-2z}{\cancel{1+z^2}}}{\frac{2+2z^2+1-z^2}{\cancel{1+z^2}}} \cdot \frac{dz}{1+z^2} = \int \frac{2z^2-2z+2}{z^2+3} \cdot \frac{dz}{1+z^2}$$

$$= \int \frac{(2z^2-2z+2)\,dz}{(z^2+3)(1+z^2)}$$

$$\frac{2z^2-2z+2}{\cancel{(z^2+3)\cdot(z^2+1)}} = \frac{Az+B}{z^2+3} + \frac{Cz+D}{z^2+1}$$

$$= \frac{(Az+B)(z^2+1) + (Cz+D)\cdot(z^2+3)}{\cancel{(z^2+3)(z^2+1)}}$$

$$2z^2-2z+2 = Az^3+Az+Bz^2+B+Cz^3+3Cz+Dz^2+3D$$

$$\Leftrightarrow \begin{cases} A + C = 0 \\ B + D = 2 \\ A + 3C = -2 \\ B + 3D = 2 \end{cases}$$

$$\begin{cases} A + C = 0 \\ A + 3C = -2 \end{cases}$$

$$\begin{cases} B + D = 2 \\ B + 3D = 2 \end{cases} \qquad D = 2 - B$$

$$\begin{cases} A + C = 0 \\ A + 3C = -2 \end{cases} \longrightarrow A = -C$$

$$-C + 3C = -2$$

$$C = -1 \Longrightarrow A = +1$$

$$B + 3\cdot(2 - B) = 2$$

$$B + 6 - 3B = 2$$

$$-2B = -4$$

$$B = 2$$

$$D = 2 - B = 2 - 2$$

$$D = 0$$

Assim, obtemos:

$$\frac{2z^2 - 2z + 2}{(z^2+3)(z^2+1)} = \frac{z+2}{z^2+3} + \frac{-z + 0}{z^2+1}$$

$$\Rightarrow \int \frac{2z^2 - 2z + 2}{(z^2+3)(z^2+1)}\, dz = \int \frac{z+2}{z^2+3}\, dz - \int \frac{z}{z^2+1}\, dz =$$

$$= \underbrace{\frac{1}{2}\int \frac{2z}{z^2+3}\, dz}_{\int \frac{du}{u}} + 2\int \frac{dz}{\underbrace{z^2+3}_{z^2+(\sqrt{3})^2}} - \frac{1}{2}\int \frac{2z}{z^2+1}\, dz$$

$u = z^2+3$

$du = 2z\,dz$

$$= \frac{1}{2}\ln(z^2+3) + 2\cdot\frac{1}{\sqrt{3}}\cdot\arctan\left(\frac{z}{\sqrt{3}}\right) - \frac{1}{2}\ln(z^2+1) + C$$

$$= \frac{1}{2}\ln\left(\tan^2\frac{x}{2}+3\right) + \frac{2}{\sqrt{3}}\arctan\left(\frac{\tan\frac{x}{2}}{\sqrt{3}}\right) - \frac{1}{2}\ln\left(\tan^2\frac{x}{2}+1\right) + C$$

$z = \tan\frac{x}{2}$