CÁLCULO II

02/07/25 - AULA 15

Seguindo o estudo de integrais por decomposição em frações parciais, vimos $\int \frac{P(x)}{q(x)}\,dx$;

onde $q(x)=0$ possui raízes reais e distintas (caso 1)

Vejamos mais um exemplo, da lista 05:

2. Calcule cada integral indefinida a seguir, comprovando a resposta indicada ao lado.

(a) $\displaystyle\int \frac{(4x-2)dx}{x^3-x^2-2x} = \ln\frac{|x^2-2x|}{(x+1)^2} + c$

$$x^3-x^2-2x=0 \iff x(x^2-x-2)=0$$

$$\iff \begin{cases} x=0 \\ x^2-x-2=0 \end{cases}$$

$$\hookrightarrow x = \frac{1\pm\sqrt{1+8}}{2}$$

$$x=\frac{1\pm 3}{2} \quad \begin{matrix} \nearrow x=2 \\ \searrow x=-1. \end{matrix}$$

Assim; $x^3-x^2-2x = x(x-2)(x+1)$.

Então:

$$\frac{4x-2}{x(x-2)(x+1)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+1}$$

$$= \frac{A(x-2)(x+1) + Bx(x+1) + Cx(x-2)}{x(x-2)(x+1)}$$

$$= \frac{Ax^2 - Ax - 2A + Bx^2 + Bx + Cx^2 - 2Cx}{x(x-2)(x+1)}$$

$0x^2$ $\quad 4x - 2 = (A+B+C)\,x^2 + (-A+B-2C)\,x - 2A.$

$$\begin{cases} A+B+C=0 \\ -A+B-2C=4 \\ -2A=-2 \end{cases} \longrightarrow \boxed{A=1}$$

$$\begin{cases} B+C=-1 \\ B-2C=5 \end{cases}$$

$\Downarrow$

$B = 5+2C$

$\Downarrow$

$5+2C+C=-1$

$3C=-6$

$\boxed{C=-2}$

$B = 5+2C$

$B = 5+2\cdot(-2) \Rightarrow \boxed{B=1}$

Assim, obtemos:

$$\frac{4x-2}{x^3-x^2-2x} = \frac{1}{x} + \frac{1}{x-2} - \frac{2}{x+1}$$

$$\Rightarrow \int \frac{(4x-2)dx}{x^3-x^2-2x} = \int \left(\frac{1}{x} + \frac{1}{x-2} - \frac{2}{x+1} \right) dx$$

$$= \int \frac{dx}{x} + \int \frac{dx}{x-2} - 2 . \int \frac{dx}{x+1}$$

$$\boxed{\int \frac{du}{u} = \ln|u| + c}$$

$$= \ln|x| + \ln|x-2| - 2 . \ln|x+1| + c$$

$$= \ln |x| . |x-2| - \ln |x+1|^2 + c$$

$$= \ln \frac{|x(x-2)|}{|x+1|^2} + c = \ln \frac{|x^2-2x|}{(x+1)^2} + c .$$

CASO 2: $q(x)$ possui raízes reais repetidas.

Neste caso, tem-se: raízes α_1 repetindo m_1 vezes,

α_2 repetindo m_2 vezes

α_3 repetindo m_3 vezes

$\vdots$

α_k repetindo m_k vezes.

Então:

$$f(x) = (x-\alpha_1)^{m_1} \cdot (x-\alpha_2)^{m_2} \cdots (x-\alpha_k)^{m_k}$$

Assim:

$$\frac{P(x)}{q(x)} = \frac{A_1}{(x-\alpha_1)^{m_1}} + \frac{A_2}{(x-\alpha_1)^{m_1-1}} + \cdots + \frac{A_{m_1}}{x-\alpha_1} +$$

$$+ \frac{B_1}{(x-\alpha_2)^{m_2}} + \frac{B_2}{(x-\alpha_2)^{m_2-1}} + \cdots + \frac{B_{m_2}}{x-\alpha_2} +$$

$$\cdots + \frac{W_1}{(x-\alpha_k)^{m_k}} + \frac{W_2}{(x-\alpha_k)^{m_k-1}} + \cdots + \frac{W_{m_k}}{x-\alpha_k}.$$

Vejamos exemplos:

$$\int \frac{(x^3-2x)\,dx}{(x^2-1)(x+2)^2}$$

$$(x^2-1)(x+2)^2 = \underbrace{(x+1)(x-1)}_{\text{CASO 1}} \cdot \underbrace{(x+2)^2}_{\text{CASO 2}}$$

Então, deveremos ter a decomposição:

$$\frac{x^3-2x}{(x^2-1)(x+2)^2} \equiv \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x+2)^2} + \frac{D}{x+2}$$

$$\equiv \frac{A(x-1)(x+2)^2 + B(x+1)(x+2)^2 + C(x^2-1) + D(x^2-1)(x+2)}{(x+1)(x-1)(x+2)^2}$$

$$x^3-2x \equiv (Ax-A)(x^2+4x+4) + (Bx+B)(x^2+4x+4) +$$
$$+ Cx^2 - C + (Dx^2-D)(x+2)$$

$$x^3-2x \equiv Ax^3 + 4Ax^2 + \cancel{4Ax} - Ax^2 - \cancel{4Ax} - 4A + Bx^3$$
$$+ 4Bx^2 + 4Bx + Bx^2 + 4Bx + 4B + Cx^2 - C +$$
$$+ Dx^3 + 2Dx^2 - Dx - 2D$$

$$\begin{cases} A + B + D = 1 \\ 3A + 5B + C + 2D = 0 \\ 8B - D = -2 \\ -4A + 4B - C - 2D = 0 \end{cases}$$

$$18B = -1 \Rightarrow \boxed{B = -\frac{1}{18}}$$

$$8 \cdot \left(-\frac{1}{18}\right) - D = -2$$

$$-D = -2 + \frac{4}{9}$$

$$D = 2 - \frac{4}{9}$$

$$\boxed{D = \frac{14}{9}}$$

$$A + \left(-\frac{1}{18}\right) + \frac{14}{9} = 1$$

$$\frac{27}{18} = \frac{\cancel{3} \cdot 3}{3 \cdot 2 \cdot \cancel{3}} = \frac{3}{2}$$

$$A = 1 - \frac{27}{18} = 1 - \frac{3}{2} = -\frac{1}{2}$$

$$\Rightarrow \boxed{A = -\frac{1}{2}}$$

Na quarta equação determinaremos o valor de C:

$$-4A + 4B - C - 2D = 0$$

$$\therefore\ C = -4A + 4B - 2D$$

$$C = -4\cdot\left(-\frac{1}{2}\right) + 4\cdot\left(-\frac{1}{18}\right) - 2\cdot\left(\frac{14}{9}\right)$$

$$C = 2 - \frac{2}{9} - \frac{28}{9} = 2 - \frac{30}{9} = 2 - \frac{\cancel{3}\cdot 10}{\cancel{9}_{3}}$$

$$= 2 - \frac{10}{3} = -\frac{4}{3} \Rightarrow \boxed{C = -\frac{4}{3}}$$

Portanto, obtemos

$$\frac{x^3 - 2x}{(x^2-1)(x+2)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x+2)^2} + \frac{D}{x+2}$$

$$= \frac{-\frac{1}{2}}{x+1} + \frac{-\frac{1}{18}}{x-1} + \frac{-\frac{4}{3}}{(x+2)^2} + \frac{\frac{14}{9}}{x+2}$$

Disso, obtemos:

$$\int \frac{(x^3-2x)dx}{(x^2-1)(x+2)^2} = -\frac{1}{2}\int\frac{dx}{x+1} - \frac{1}{18}\int\frac{dx}{x-1} - \frac{4}{3}\int\frac{dx}{(x+2)^2} +$$

$$+ \frac{14}{9}\int\frac{dx}{x+2}$$

$$= -\frac{1}{2}\int \frac{dx}{x+1} - \frac{1}{18}\int \frac{dx}{x-1} - \frac{4}{3}\int (x+2)^{-2}\,dx + \frac{14}{9}\int \frac{dx}{x+2}$$

$\int u^{r}\,du$

$\int \frac{du}{u} = \ln|u| + c$

$$= -\frac{1}{2}\ln|x+1| - \frac{1}{18}\ln|x-1| - \frac{4}{3}\,\frac{(x+2)^{-1}}{-2} + \frac{14}{9}\ln|x+2| + c$$

$$= -\frac{1}{2}\ln|x+1| - \frac{1}{18}\ln|x-1| + \frac{14}{9}\ln|x+2| + \frac{4}{3}\,\frac{1}{x+2} + c$$

02) $\int \frac{(x-1)\,dx}{(x+3)^2(x+1)} = ?$

$$\frac{x-1}{(x+3)^2(x+1)} = \frac{A}{(x+3)^2} + \frac{B}{x+3} + \frac{C}{x+1}$$

$$= \frac{A(x+1) + B(x+3)(x+1) + C(x+3)^2}{(x+3)^2(x+1)}$$

Disso:

$$1x - 1 = Ax + A + Bx^2 + 4Bx + 3B + Cx^2 + 6Cx + 9C$$

$$\begin{cases} B + C = 0 \\ A + 4B + 6C = 1 \\ A + 3B + 9C = -1 \end{cases}$$

$$- \begin{cases} \cancel{A} + 4B + 6C = 1 \\ -\cancel{A} - 3B - 9C = +1 \end{cases}$$

$$B - 3C = 2$$

$$\begin{cases} B + C = 0 \rightarrow B = -C \\ B - 3C = 2 \end{cases}$$

$$-C - 3C = 2 \Rightarrow \boxed{C = -\frac{1}{2}}$$

$$\boxed{B = +\frac{1}{2}}$$

$$A + 4B + 6C = 1$$

$$A + 4 \cdot \left(\frac{1}{2}\right) + 6 \cdot \left(-\frac{1}{2}\right) = 1$$

$$A + 2 - 3 = 1 \Rightarrow \boxed{A = 2}$$

Disso, obtemos:

$$\frac{x-1}{(x+3)^2(x+1)} = \frac{2}{(x+3)^2} + \frac{\frac{1}{2}}{x+3} + \frac{-\frac{1}{2}}{x+1}$$

$$\rightarrow \int \frac{(x-1)\,dx}{(x+3)^2(x+1)} = 2\int (x+3)^{-2}dx + \frac{1}{2}\int \frac{dx}{x+3} - \frac{1}{2}\int \frac{dx}{x+1}$$

$\int r^k dr$

$r = x+3$

$dr = dx$

$\int \frac{dr}{r}$

$$= 2\,\frac{(x+3)^{-1}}{-1} + \frac{1}{2}\ln|x+3| - \frac{1}{2}\ln|x+1| + C$$

$$= \frac{-2}{x+3} + \frac{1}{2}\ln\left|\frac{x+3}{x+1}\right| + c$$

Lista 05.

02)

$$(\ell)\ \int \frac{x^2dx}{(x^2-1)^3} = \frac{1}{16}\left[-\frac{1}{x-1} - \ln|x-1| + \ln|x+1| + \frac{1}{(x+1)^2} - \frac{1}{(x+1)} - \frac{1}{(x-1)^2}\right] + c$$

Note que: $(x^2-1)^3 = [(x+1)(x-1)]^3 = (x+1)^3(x-1)^3$

Assim, teremos:

$$\frac{x^2}{(x^2-1)^3} = \frac{A}{(x+1)^3} + \frac{B}{(x+1)^2} + \frac{C}{x+1} + \frac{D}{(x-1)^3} + \frac{E}{(x-1)^2} + \frac{F}{x-1}.$$

... et cetera ...

CASO 3: $q(x)$ possui raízes complexas e distintas.

É semelhante ao caso 1, mas na decomposição haverão fatores quadráticos irredutíveis no denominador. Então, deve-se ter polinômio de grau 1 no numerador correspondente.

Ex.:

(a) $$\int \frac{x\,dx}{(x+1)(x^2+4)}$$

$x^2+4=0 \Leftrightarrow x^2=-4$

Faz-se a decomposição:

$$\frac{x}{(x+1)(x^2+4)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+4}$$

$$= \frac{A(x^2+4) + (Bx+C)(x+1)}{(x+1)(x^2+4)}$$

$$x = Ax^2 + 4A + Bx^2 + Bx + Cx + C$$

$$\begin{cases} A + B = 0 \rightarrow A = -B \\ B + C = 1 \\ 4A + C = 0 \end{cases}$$

$$\begin{cases} B + C = 1 \rightarrow C = 1 - B \\ -4B + C = 0 \end{cases}$$

$$-4 \cdot B + 1 - B = 0$$

$$-5B = -1 \Rightarrow \boxed{B = \frac{1}{5}}$$

$$\Rightarrow A = -B \Rightarrow \boxed{A = -\frac{1}{5}}$$

$$C = 1 - B \Rightarrow C = 1 - \frac{1}{5} = \frac{4}{5} \Rightarrow \boxed{C = \frac{4}{5}}$$

Disso, obtemos:

$$\frac{x}{(x+1)(x^2+4)} = \frac{-\frac{1}{5}}{x+1} + \frac{\frac{1}{5}x + \frac{4}{5}}{x^2+4}$$

$$\Rightarrow \int \frac{x\,dx}{(x+1)(x^2+4)} = -\frac{1}{5}\int \frac{dx}{x+2} + \frac{1}{5}\int \frac{x+4}{x^2+4}\,dx$$

$$\to \int \frac{du}{u^2+a^2} = \frac{1}{a}\arctan\frac{u}{a}$$

$$= -\frac{1}{5}\int \frac{dx}{x+1} + \frac{1}{2}\cdot\frac{1}{5}\int \underbrace{\frac{2x\,dx}{x^2+4}}_{\int \frac{du}{u}} + \frac{4}{5}\int \frac{dx}{x^2+4}$$

$$u = x^2+4 \to du = 2x\,dx$$

$$= -\frac{1}{5}\ln|x+1| + \frac{1}{10}\ln|x^2+4| + \frac{4}{5}\cdot\frac{1}{2}\cdot\arctan\left(\frac{x}{2}\right) + C$$

$$= -\frac{1}{5}\cdot\ln|x+1| + \frac{1}{10}\ln(x^2+4) + \frac{2}{5}\cdot\arctan\left(\frac{x}{2}\right) + C$$

(b) Vamos apenas decompor o seguinte exemplo:

$$\int \frac{(x^3-x+1)\,dx}{(x-3)^2(x^2+1)}.$$

solução:

$$\frac{x^3 - x + 1}{(x-3)^2(x^2+1)} = \frac{A}{(x-3)^2} + \frac{B}{x-3} + \frac{Cx+D}{x^2+1}$$

$$\frac{A(x^2+1) + B\cdot(x-3)(x^2+1) + (Cx+D)(x-3)^2}{(x-3)^2(x^2+1)}$$

etc...