VARIÁVEIS COMPLEXAS

11/06/25 — AULA 13

AULA DE EXERCÍCIOS

LISTA 05

7. Verifique a convergência ou divergência de cada série complexa a seguir:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(2i)^n}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(in)^n}$ (c) $\displaystyle\sum_{n=1}^{\infty} e^{in}$

Pelo teste da razão, sendo $a_n = \dfrac{n!}{i^n \cdot n^n}$;

temos:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} =$$

$$= \lim_{n\to\infty} \frac{\dfrac{(n+1)!}{\underbrace{|i|^{n+1}}_{1=} \cdot (n+1)^{n+1}}}{\dfrac{n!}{\underbrace{|i|^n}_{1=} \cdot n^n}} =$$

$$= \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \lim_{n\to\infty} \frac{\cancel{(n+1)} \cdot \cancel{n!} \cdot n^n}{(n+1)^n \cdot \cancel{(n+1)} \cdot \cancel{n!}}$$

$$= \lim_{n\to\infty} \frac{n^n}{(n+1)^n} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n =$$

1^∞

$$= \lim_{n\to\infty} \left(1 + \frac{n}{n+1} - 1\right)^n = \lim_{n\to\infty} \left(1 + \frac{\cancel{n} - \cancel{n} - 1}{n+1}\right)^n =$$

$$= \lim_{n\to\infty} \left(1 + \frac{-1}{n+1}\right)^n = \left[\underbrace{\lim_{n\to\infty}\left(1 + \frac{-1}{n+1}\right)}_{e}\right]^{-\frac{n+1}{1}\cdot\left(\frac{-1}{n+1}\right) n}$$

$$= e^{\lim_{n\to\infty} \frac{-n}{n+1}} = e^{-1} = \frac{1}{e} < 1$$

Logo, a série converge absolutamente.

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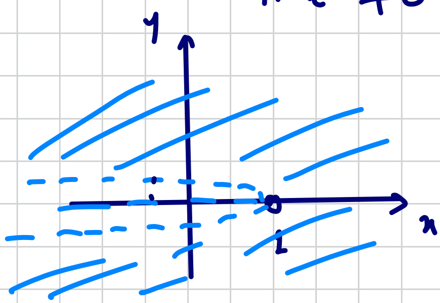
1. Determine a parte real $u(x, y)$ e a parte imaginária $v(x, y)$ de cada função complexa $w = f(z)$ abaixo.

(a) $w = \dfrac{z+2}{z-2}$ (b) $w = \dfrac{z^2}{\overline{z}-1}$ (c) $w = \operatorname{sen} z$

(d) $w = \cos \dfrac{1}{z}$ (e) $w = \log(1-z)$ [no ramo principal]

(e) $w = \log(1-z)$ (no ramo principal).

$$1 - z \neq 0 \iff z \neq 1.$$



Definimos $-\pi < \arg(1-z) < \pi$.

$$1 - z = 1 - (x+iy) = (1-x) - iy$$

Disso; $|1-z| = \sqrt{(1-x)^2 + y^2}$

$$\arg(1-z) = \arctan\left(\frac{-y}{1-x}\right)$$

Assim:

$$\log(1-z) = \ln|1-z| + i\cdot\left(\arg(1-z) + 2k\pi\right)$$

$$= \ln\sqrt{(1-x)^2+y^2} + i\cdot\left(\arctan\left(\frac{-y}{1-x}\right) + 2k\pi\right)$$

No ramo principal $(k=0)$, teremos:

$$\log(1-z) = \underbrace{\frac{1}{2}\ln\left((1-x)^2+y^2\right)}_{u(x,y)} + i\cdot\underbrace{\arctan\left(\frac{y}{x-1}\right)}_{v(x,y)}$$

Disso, vem:

$$\Rightarrow \begin{cases} u(x,y) = \frac{1}{2}\ln\left((1-x)^2+y^2\right) \\ v(x,y) = \arctan\left(\frac{y}{x-1}\right) \end{cases}$$

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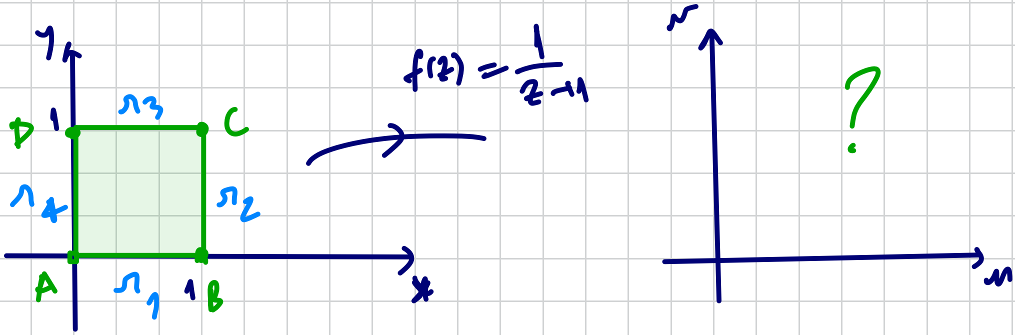
9. Um quadrado no plano complexo tem vértices em $A(0,0)$, $B(1,0)$, $C(1,1)$ e $D(0,1)$. Determine a região do plano w na qual o quadrado é transformado por

(a) $f(z) = z^2$ (b) $f(z) = iz - 1$ (c) $f(z) = \dfrac{1}{z+1}$

(c)



$$f(z) = \frac{1}{z+1} = \frac{1}{x+iy+1} = \frac{1}{(x+1)+iy}$$

$$= \frac{1}{(x+1)+iy} \cdot \frac{(x+1)-iy}{(x+1)-iy} = \frac{(x+1)-iy}{(x+1)^2+y^2}$$

$$\Rightarrow \begin{cases} u(x,y) = \dfrac{x+1}{(x+1)^2+y^2} \\[2ex] v(x,y) = \dfrac{-y}{(x+1)^2+y^2} \end{cases}$$

$$u^2 + v^2 = \frac{(x+1)^2}{[(x+1)^2+y^2]^2} + \frac{y^2}{[(x+1)^2+y^2]^2}$$

$$u^2 + v^2 = \frac{\cancel{(x+1)^2 + y^2}}{[(x+1)^2+y^2]^{\cancel{2}}}$$

$$u^2 + v^2 = \frac{1}{(x+1)^2+y^2}$$

Vamos observar cada segmento $r_1, r_2, r_3\ r_4$ do quadrado:

$r_1 : y=0$ e $0 \leq x \leq 1$. neste caso:

$$1 \leq x+1 \leq 2 \iff 1 \leq (x+1)^2 \leq 4$$

$$\text{e} \quad y=0 .$$

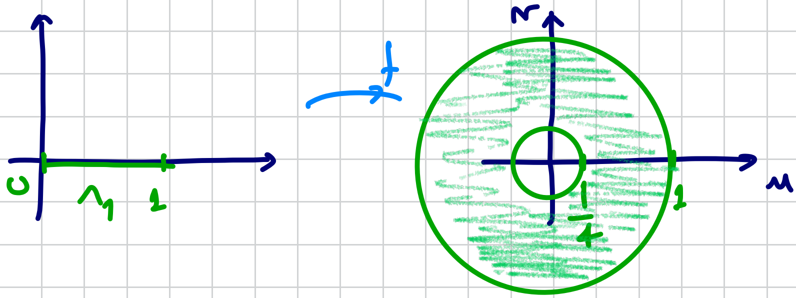
Assim $1 \leq (x+1)^2 + y^2 \leq 4$

$$\Rightarrow \frac{1}{4} \leq \frac{1}{(x+1)^2+y^2} \leq 1 .$$

Ou seja o segmento r_1 se transforma

em circunferências

$$u^2+v^2 \le a^2 \;; \text{ onde } \frac{1}{4} \le a^2 \le 1.$$



r_3: $y = 1$; e $0 \le x \le 1$.

$y^2 = 1$

$$\Rightarrow 1 \le x+1 \le 2$$

$$\Rightarrow 1 \le (x+1)^2 \le 4.$$

$$\Rightarrow 2 \le (x+1)^2 + y^2 \le 5$$

$$\frac{1}{5} \le \frac{1}{(x^2+1)^2+y^2} \le \frac{1}{2}$$

ou seja: $u^2+v^2 = a^2$, com $\frac{1}{5} \le a^2 \le \frac{1}{2}$

Ω_2: $x = 1$ e $0 \leq y \leq 1$.

$$\left.\begin{array}{l} 0 \leq y \leq 1 \\ (x+1)^2 = 2^2 = 4 \end{array}\right\} \quad \begin{array}{l} 0 \leq y^2 \leq 1. \\ \to 4 \leq (x+1)^2 + y^2 \leq 5 \end{array}$$

$$\Rightarrow \quad u^2 + v^2 = \frac{1}{(x+1)^2 + y^2} \qquad \frac{1}{5} \leq \frac{1}{(x+1)^2 + y^2} \leq \frac{1}{4}$$

(. . .) ?

12. Escreva os seguintes números complexos na forma algébrica:

(a) e^{2+i} (b) $\operatorname{sen}(1+i)$ (c) $\log(2^i)$

(c) $\log(2^i)$

$$\log(2^i) = \ln|2^i| + i\left[\arg\left((2^i) + 2k\pi\right)\right];$$

onde:

$$2^i = e^{i \cdot \ln 2} = \cos(\ln 2) + i \cdot \operatorname{sen}(\ln 2)$$

$$\Rightarrow |2^i| = 1.$$

$$\arg(2^i) = \arctan\left(\frac{\operatorname{sen}(\ln 2)}{\cos(\ln 2)}\right)$$

$$= \arctan(\tan(\ln 2)) =$$

$$= \ln 2$$

Assim:

$$\log 2^i = \underbrace{\ln|1|}_{=0} + i\cdot(\ln 2 + 2k\pi)$$

$$= \boxed{0 + i\cdot(\ln 2 + 2k\pi)}$$

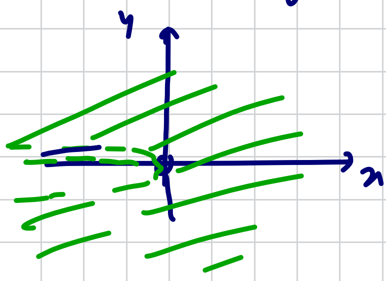
25. Em cada caso abaixo, encontre um ramo que torne a função plurívoca dada em unívoca.

(a) $f(z) = \sqrt{1-z}$ (b) $f(z) = \log\log(z)$

(c) $f(z) = \sqrt{z(z-1)}$ (d) $f(z) = \sqrt{z^2+2z-3}$

(b) $f(z) = \log(\log z)$.

$$\log z = \ln|z| + i\cdot(\arg z + 2k\pi)$$



$k=0$

(Ramo principal)

$$\Rightarrow \log(\log z) = \log(\underbrace{\ln|z| + i\cdot\arg z}_{})$$

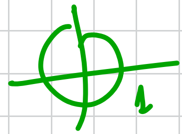
$\ln|z| = 0 \Leftrightarrow |z| = 1.$

$\Downarrow$

neste caso, teremos

$$\log(\log z) = \log(\underbrace{i \cdot \arg z}_{\Downarrow})$$

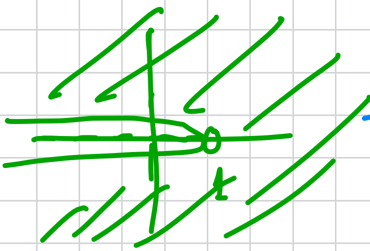
0 se, e só se $\arg z = 0$



lembre que tomamos o ramo $-\pi < \arg z \leq \pi$

$\Downarrow$

conclusão, devemos ter $z \neq 1$.



$-\pi < \arg(\log z) \leq \pi$

conclusão, $f(z) = \log(\log z)$ fica unívoca em $\mathbb{C} \setminus \{x+iy : y=0 \text{ e } x<1\}$