CÁLCULO II

11/06/25 – AULA 13

AULA DE EXERCÍCIOS: (LISTA 04)

L4:

1. Calcule as integrais a seguir:

(a) $\displaystyle\int \frac{(2x+3)dx}{x^2-4x+1}$ (b) $\displaystyle\int \frac{(7x-4)dx}{3x^2-4x+5}$ (c) $\displaystyle\int \frac{(2x-4)dx}{\sqrt{x^2-12x+4}}$

(d) $\displaystyle\int \frac{(3x-1)dx}{\sqrt{x^2+x+1}}$ (e) $\displaystyle\int \frac{(1-3x)dx}{4x^2+12x-5}$ (f) $\displaystyle\int \frac{(2x+3)dx}{\sqrt{2-3x^2}}$

(c) $$\int \frac{(2x-4)dx}{\sqrt{x^2-12x+4}} = ?$$

$$v = x^2 - 12x + 4$$

$$dv = (2x-12)\,dx$$

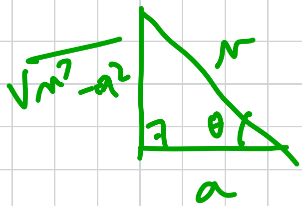
$$\int \frac{(2x-4)\,dx}{\sqrt{x^2-12x+4}} = \int \frac{(2x-12+8)\,dx}{\sqrt{x^2-12x+4}} =$$

$$\int \frac{(2x-12)\,dx}{\sqrt{x^2-12x+4}} + \int \frac{8\,dx}{\sqrt{x^2-12x+4}} =$$

$$= \int \underbrace{(x^2 - 12x + 4)^{-\frac{1}{2}} \cdot (2x - 12)\,dx} + 8 \int \frac{dx}{\sqrt{(x-6)^2 - 32}} =$$

$$\int r^k\,dr = \frac{r^{k+1}}{k+1} + c$$

$$\int \frac{dr}{\sqrt{r^2 - a^2}}$$



$$\tan\theta = \frac{\sqrt{r^2 - a^2}}{a}$$

$$\Rightarrow \boxed{\sqrt{r^2 - a^2} = a \cdot \tan\theta}.$$

$$\cos\theta = \frac{a}{r} \Rightarrow r = a \cdot \sec\theta.$$

$$\boxed{dr = a \cdot \sec\theta \cdot \tan\theta\,d\theta}$$

$$\Rightarrow \int \frac{dr}{\sqrt{r^2 - a^2}} = \int \frac{\cancel{a}\sec\theta \cdot \cancel{\tan\theta}\,d\theta}{\cancel{a} \cdot \cancel{\tan\theta}} = \int \sec\theta\,d\theta =$$

$$= \ln|\sec\theta + \tan\theta| + c$$

$$= \ln\left| \frac{u}{a} + \frac{\sqrt{u^2-a^2}}{a} \right| + c$$

$$= \ln\left| u + \sqrt{u^2-a^2} \right| + c$$

$u = x-6$
$du = dx$
$a = \sqrt{32}$

$$= \frac{(x^2-12x+4)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + 8 \cdot \ln\left| x-6 + \sqrt{(x-6)^2 - (\sqrt{32})^2} \right| + c$$

$$= 2 \cdot \sqrt{x^2-12x+4} + 8 \cdot \ln\left| x-6 + \sqrt{x^2-12x+4} \right| + c.$$

(e) $\displaystyle\int \frac{(1-3x)\,dx}{4x^2+12x-5} = \int \frac{-\frac{3}{8}(8x+12) + \alpha}{4x^2+12x-5} =$

$u = 4x^2+12x-5$

$\hookrightarrow du = (8x+12)\,dx$

$$-3x - \frac{3 \cdot \cancel{4} \cdot 3}{\cancel{8}_{2}} + \alpha$$

$$-3x - \frac{9}{2} + \alpha = -3x + 1$$

$$-\frac{9}{2}+\alpha=1$$

$$\alpha=1+\frac{9}{2}$$

$$\alpha=\frac{11}{2}$$

$$= \int \frac{-\frac{3}{8}(8x+12)+\frac{11}{2}}{4x^2+12x-5}\,dx =$$

$$= -\frac{3}{8}\int \frac{(8x+12)\,dx}{4x^2+12x-5} + \frac{11}{2}\int \frac{dx}{4x^2+12x-5} =$$

$$\int \frac{dv}{v} = \ln|v|+C$$

COMPLETAR UM QUADRADO.

$$4x^2+12x-5 = 4\cdot\left[(x+k)^2+\ell\right]$$

$$= 4\cdot\left(x^2+2kx+k^2+\ell\right)$$

$$= 4x^2+8kx+4k^2+4\ell$$

$$\Leftrightarrow \quad 8k = 12 \quad \Rightarrow \quad k = \frac{12}{8} = \frac{3\cdot\cancel{4}}{\cancel{4}\cdot 2} = \frac{3}{2}$$

$$4k^2 + 4l = -5$$

$$\cancel{4}\cdot\left(\frac{9}{\cancel{4}}\right) + 4l = -5$$

$$4l = -5 - 9$$

$$4l = -14$$

$$l = -\frac{7}{2}$$

Assim:

$$4x^2 + 12x - 5 = 4\left[\left(x + \frac{3}{2}\right)^2 - \frac{7}{2}\right];$$

e então:

$$= -\frac{3}{8}\ln|4x^2+12x-5| + \frac{11}{2}\cdot\int \frac{dx}{4\cdot\left[\left(x+\frac{3}{2}\right)^2 - \frac{7}{2}\right]}$$

$$= -\frac{3}{8}\cdot\ln|4x^2+12x-5| + \frac{11}{8}\int \frac{dx}{\left(x+\frac{3}{2}\right)^2 - \left(\sqrt{\frac{7}{2}}\right)^2} =$$

$$\int \frac{du}{u^2 - a^2} = \frac{1}{a}\ln\left|\frac{u-a}{\sqrt{u^2-a^2}}\right| + C$$

$u = x + \frac{3}{2} \Rightarrow du = dx.$

$$= -\frac{3}{8} \cdot \ln|4x^2+12x-5| + \frac{11}{8} \cdot \frac{1}{\sqrt{\frac{7}{2}}} \cdot \ln\left|\frac{x+\frac{3}{2}-\sqrt{\frac{7}{2}}}{\sqrt{(x+\frac{3}{2})^2-\frac{7}{2}}}\right| + C$$

$$= -\frac{3}{8} \cdot \ln|4x^2+12x-5| + \frac{11}{8} \cdot \sqrt{\frac{2}{7}} \ln\left|\frac{x+\frac{3}{2}+\sqrt{\frac{7}{2}}}{\sqrt{(x+\frac{3}{2})^2-\frac{7}{2}}}\right| + C$$

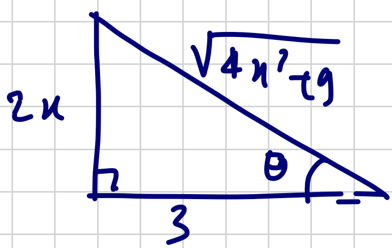
L4:

2. Calcule as integrais a seguir, através de uma substituição trigonométrica:

(a) $\int \frac{dx}{9x^2+5}$ (b) $\int \frac{dx}{\sqrt{x^2-16}}$ (c) $\int \frac{dx}{4x^2+4x-6}$

(d) $\int \sqrt{4x^2+9}\,dx$ (e) $\int \frac{dx}{\sqrt{1+2x+3x^2}}$ (f) $\int \frac{\sqrt{x^2+9}}{x^3}dx$

(d) $\int \sqrt{4x^2+9}\, dx$



$$\cos\theta = \frac{3}{\sqrt{4x^2+9}} \Rightarrow \boxed{\sqrt{4x^2+9} = 3\cdot\sec\theta.}$$

$$\tan\theta = \frac{2x}{3} \Rightarrow \boxed{x = \frac{3}{2}\tan\theta.}$$

$$\boxed{dx = \frac{3}{2}\cdot\sec^2\theta\, d\theta.}$$

Assim, obtemos:

$$\int \sqrt{4x^2+9}\, dx = \int 3\cdot\sec\theta\cdot\frac{3}{2}\sec^2\theta\, d\theta =$$

$$= \frac{9}{2}\int \sec^2\theta\cdot\sec\theta\, d\theta =$$

INTEGRAL POR PARTES (FINAL DA AULA 11, EXEMPLO f.)

$$= \frac{9}{2}\cdot\left[\frac{1}{2}\ln|\sec\theta+\tan\theta| + \frac{1}{2}\tan\theta\cdot\sec\theta + C\right] =$$

$$= \frac{9}{4}\ln|\sec\theta + \tan\theta| + \frac{9}{2}\tan\theta\sec\theta + C =$$

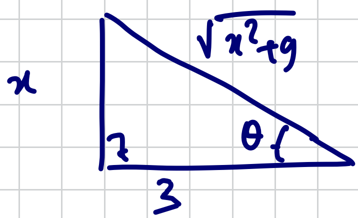
$$\sec\theta = \frac{\sqrt{4x^2+9}}{3}$$

$$\tan\theta = \frac{2x}{3}$$

$$= \frac{9}{4}\cdot\ln\left|\frac{\sqrt{4x^2+9}}{3} + \frac{2x}{3}\right| + \frac{\cancel{9}}{\cancel{2}}\,\frac{\cancel{2}x}{\cancel{3}}\cdot\frac{\sqrt{4x^2+9}}{\cancel{3}} + C$$

$$= \boxed{\frac{9}{4}\cdot\ln\left|\sqrt{4x^2+9} + 2x\right| + x\cdot\sqrt{4x^2+9} + C}$$

(f) $\displaystyle\int \frac{\sqrt{x^2+9}}{x^3}\,dx$



$$\tan\theta = \frac{x}{3} \Rightarrow x = 3\cdot\tan\theta.$$

$$\Rightarrow dx = 3\cdot\sec^2\theta\,d\theta.$$

$$\cos\theta = \frac{3}{\sqrt{x^2+9}} \Rightarrow \sqrt{x^2+9} = 3\cdot\sec\theta.$$

Assim, vamos encontrar:

$$\int \frac{\sqrt{x^2+9}}{x^3}\,dx = \int \frac{3\cdot\sec\theta}{(3\cdot\tan\theta)^3}\,3\cdot\sec^2\theta\,d\theta =$$

$$= \int \frac{\cancel{3^2}}{3^{\cancel{3}}}\cdot\frac{\sec^3\theta}{\tan^3\theta}\,d\theta = \frac{1}{3}\cdot\int \frac{1}{\cancel{\cos^3\theta}}\cdot\frac{\cancel{\cos^3\theta}}{\operatorname{sen}^3\theta}\,d\theta =$$

$$= \frac{1}{3}\cdot\int \csc^3\theta\,d\theta = \underbrace{\frac{1}{3}\cdot\int \csc^2\theta\cdot\csc\theta\,d\theta}_{} =$$

$$= \frac{1}{3}\cdot\int (1+\cot^2\theta)\cdot\csc\theta\,d\theta = \frac{1}{3}\underbrace{\int \csc\theta\,d\theta}_{\ln|\csc\theta-\cot\theta|+C} + \frac{1}{3}\underbrace{\int \cot^2\theta\cdot\csc\theta\,d\theta}_{\text{integrar por partes } (*)}$$

$1+\cot^2\theta=\csc^2\theta$

$(*)$ $$\int \cot^2\theta\cdot\csc\theta\,d\theta = \int u\,dv = u\cdot v - \int v\,du =$$

$u = \cot\theta \Rightarrow du = -\csc^2\theta$

$dv = \csc\theta\cdot\cot\theta \Rightarrow$

$v = \int \csc\theta\cot\theta\,d\theta = -\csc\theta$

$$= \cot\theta \cdot (-\csc\theta) - \int -\csc\theta \cdot (-\csc^2\theta)\, d\theta$$

$$= -\csc\theta \cdot \cot\theta - \int \csc^3\theta \, d\theta.$$

Arsim:

$$\frac{1}{3}\int \csc^3\theta \, d\theta = \frac{1}{3} \ln|\csc\theta - \cot\theta| + \frac{1}{3}\int \cot^2\theta \csc\theta \, d\theta$$

$$= \frac{1}{3} \ln|\csc\theta - \cot\theta| + \frac{1}{3} \cdot \left(-\csc\theta \cdot \cot\theta - \int \csc^3\theta \, d\theta\right)$$

$$= \frac{1}{3} \ln|\csc\theta - \cot\theta) - \frac{1}{3}\csc\theta \cdot \cot\theta - \frac{1}{3}\int \csc^3\theta \, d\theta$$

$$\frac{2}{3}\int \csc^3\theta \, d\theta = \frac{1}{3} \ln|\csc\theta - \cot\theta| - \frac{1}{3}\csc\theta \cot\theta + c$$

$$\Rightarrow \int \csc^3\theta \, d\theta = \frac{1}{2} \ln|\csc\theta - \cot\theta| - \frac{1}{2}\csc\theta \cdot \cot\theta + c$$

So firm:

$$\int \frac{\sqrt{x^2+9}}{x^3}\,dx = \frac{1}{3}\int \csc^3\theta\, d\theta =$$

$$= \frac{1}{3}\cdot\left(\frac{1}{2}\ln|\csc\theta-\cot\theta| - \frac{1}{2}\csc\theta\cdot\cot\theta + C\right)$$

$$= \frac{1}{6}\ln|\csc\theta-\cot\theta| - \frac{1}{6}\csc\theta\cdot\cot\theta + C$$

$$= \frac{1}{6}\ln\left|\frac{\sqrt{x^2+9}}{x} - \frac{3}{x}\right| - \frac{1}{\cancel{6}_{\,2}}\cdot\frac{\sqrt{x^2+9}}{x}\cdot\frac{\cancel{3}}{x} + C$$

$$= \frac{1}{6}\ln\left|\frac{\sqrt{x^2+9}-3}{x}\right| - \frac{\sqrt{x^2+9}}{2x^2} + C$$

L4:

9. Calcule cada integral definida a seguir:

(a) $\int_{-1}^{2} \ln(x+2)dx$ (b) $\int_{0}^{\frac{\pi}{2}} \cos\sqrt{2x}\, dx$ (c) $\int_{0}^{1} x^2 e^{-x} dx$

(a) $\underline{\underline{1^{\underline{o}}}}$: achar a antiderivada:

$$\int \ln(x+2)dx = \int u\,dv = u \cdot v - \int v \cdot du$$

$$\begin{cases} u = \ln(x+2) \Rightarrow du = \frac{1}{x+2}\, dx \\ dv = dx \Rightarrow v = \int dx \Rightarrow v = x \end{cases}$$

$$\int \ln(x+2)dx = x \cdot \ln(x+2) - \int x \cdot \frac{dx}{x+2}$$

$$= x \cdot \ln(x+2) - \int \frac{x+2-2}{x+2} dx$$

$$= x \cdot \ln(x+2) - \int \underbrace{\frac{\cancel{x+2}}{\cancel{x+2}}}_{1} dx - \int \frac{-2}{x+2} dx$$

$$= x \cdot \ln(x+2) - x + 2\int \frac{dx}{x+2}$$

$$\int \frac{du}{u}$$

$u = x+2$

$du = dx.$

ok!

$$= x\ln(x+2) - x + 2 \cdot \ln|x+2| + C$$

Por fim,

$$\int_{-1}^{2} \ln(x+2)\,dx = \left(x\ln(x+2) - x + 2\ln|x+2|\right)\Bigg|_{-1}^{2} =$$

$$= 2\cdot\ln(4) - 2 + 2\ln|4| - \left(-1\underbrace{\ln 1}_{=0} - (-1) + \underbrace{2\ln|1|}_{=0}\right)$$

$$= 2\ln 4 - 2 + 2\ln 4 - 1 = \underline{\underline{-3 + 4\ln 4}}$$