cálculo 3

27/08/24 – AULA 47

LISTA 09.j

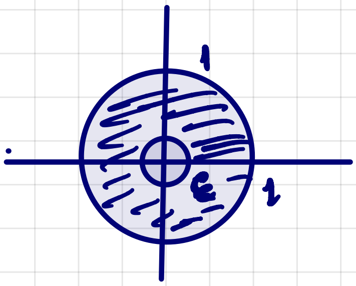
13. Calcule as integrais impróprias abaixo.

(a) $\displaystyle\int_0^\infty \int_0^\infty e^{-x-y}dxdy$ (b) $\displaystyle\int_0^\infty \int_0^\infty x^2 e^{-x^2-y^2}dxdy$

(c) $\displaystyle\iint_{x^2+y^2\leq 1} \frac{x^2dxdy}{(x^2+y^2)^{\frac{7}{4}}}$ (d) $\displaystyle\iint_{x^2+y^2\leq 1} \ln\sqrt{x^2+y^2}dxdy$

(d) $f(x,y) = \ln\sqrt{x^2+y^2}\,dx\,dy$

Note que $x^2+y^2 \neq 0$, pois $\nexists \ln 0$



Dado $\varepsilon > 0$, $\varepsilon < 1$.

$\Omega_\varepsilon = \Omega \setminus B_\varepsilon(0)$.

$\Omega = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\}$.

$$\iint_\Omega \ln\sqrt{x^2+y^2}\,dxdy := \lim_{\varepsilon\to 0} \iint_{\Omega_\varepsilon} \ln \underbrace{\sqrt{x^2+y^2}}_{\rho}\,dxdy \;(=)$$

USANDO COORDENADAS POLARES:

$$\begin{cases} \rho = \sqrt{x^2+y^2} \\ dxdy = \rho\, d\rho\, d\theta. \end{cases}$$

$$(=) \lim_{\varepsilon\to 0} \underbrace{\int_{\theta=0}^{\theta=2\pi} \int_{\rho=\varepsilon}^{\rho=1} \ln \rho \,.\, \rho\, d\rho\, d\theta}_{} . \;(=)$$

VAMOS TER QUE EFETUAR UMA INTEGRAÇÃO POR PARTES

$$\int \ln\rho \cdot \rho\, d\rho = \int u \cdot dv = u \cdot v - \int v\, du.$$

$$\left[\begin{array}{l} u = \ln\rho \Rightarrow du = \frac{1}{\rho} \cdot d\rho \\ dv = \rho\, d\rho \Rightarrow v = \frac{\rho^2}{2} \end{array}\right.$$

$$\Rightarrow \int \ln\rho \cdot \rho\, d\rho = \frac{\rho^2}{2} \cdot \ln\rho - \int \frac{\cancel{\rho^2}}{2} \cdot \frac{1}{\cancel{\rho}} \cdot d\rho$$

$$= \frac{\rho^2}{2} \ln\rho - \frac{1}{2} \int \rho\, d\rho = \frac{\rho^2}{2} \ln\rho - \frac{1}{2} \cdot \frac{\rho^2}{2} + C$$

$$= \frac{\rho^2}{2} \left(\ln\rho - \frac{1}{2} \right) + C.$$

$$\Rightarrow \lim_{\varepsilon \to 0} \int_{\theta=0}^{\theta=2\pi} \frac{\rho^2}{2} \cdot \left(\ln\rho - \frac{1}{2} \right) \Bigg|_{\rho=\varepsilon}^{1} d\theta =$$

$$= \lim_{\varepsilon \to 0} \left[\frac{1}{2} \left(\underbrace{\ln 1}_{= 0} - \frac{1}{2} \right) - \frac{\varepsilon^2}{2} \cdot \left(\ln\varepsilon - \frac{1}{2} \right) \right] \cdot 2\pi =$$

$$= \lim_{\varepsilon \to 0} 2\pi \cdot \left(\frac{1}{4} - \frac{\varepsilon^2}{2} \left(\ln\varepsilon - \frac{1}{2} \right) \right) =$$

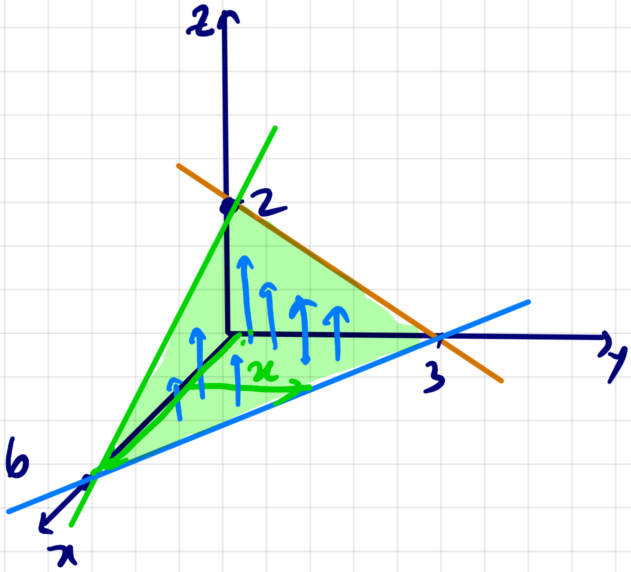
$$= \lim_{\varepsilon \to 0} 2\pi \cdot \left(\frac{1}{4} - \frac{\varepsilon^2}{2} \cdot \ln\varepsilon + \frac{1}{4} \varepsilon^2 \right) =$$

$$= \lim_{\varepsilon \to 0} 2\pi \cdot \left(\frac{1}{4} - \underbrace{\frac{\varepsilon^2}{2 \cdot \frac{1}{\ln\varepsilon}}}_{0/0} + \frac{1}{4} \varepsilon^2 \right) = 2\pi \cdot \frac{1}{4} = \frac{\pi}{2}$$

APLICA L'HOPITAL

L 10

5. Use integral tripla para calcular $\iiint_\Omega x\,dV$, onde Ω é o tetraedro limitado pelos planos $x + 2y + 3z = 6$, $x = 0$, $y = 0$ e $z = 0$.

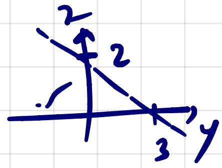


Traços nos planos coordenados:

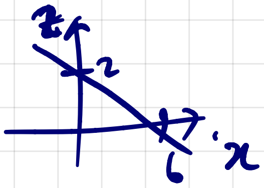
$x + 2y + 3z = 6$

* $x = 0$:

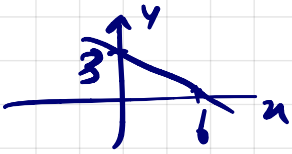
$2y + 3z = 6$



$y = 0$: $x + 3z = 6$



$z = 0$:



$x + 2y = 6$

$\Rightarrow x = 6 - 2y$

Assim, teremos:

$$\iiint_\Omega x\,dV$$

x: de zero até a reta $x = 6 - 2y$

z: de zero até o plano inclinado

$$z = \frac{6 - x - 2y}{3}$$

y: de zero até 3.

$$\iiint_{\Omega} x\,dV = \int_{y=0}^{y=3} \int_{x=0}^{x=6-2y} \int_{z=0}^{z=\frac{6-x-2y}{3}} x \, dz\,dx\,dy$$

$$= \int_{y=0}^{y=3} \int_{x=0}^{x=6-2y} x \left(\int_{z=0}^{z=\frac{6-x-2y}{3}} dz \right) dx \; dy$$

$$= \int_{y=0}^{y=3} \int_{x=0}^{x=6-2y} x \, . \; z \Big|_{z=0}^{z=\frac{6-x-2y}{3}} dx \; dy$$

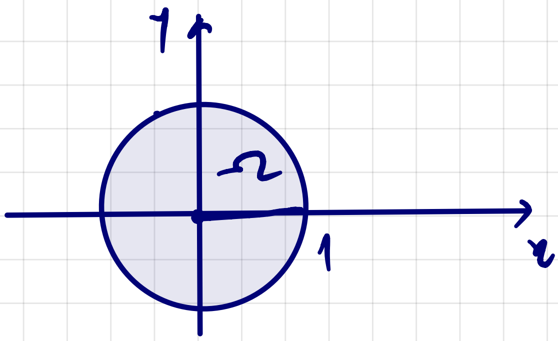
et cetera . . . .

L9 II

1. Use coordenadas polares para calcular cada integral a seguir.

(a) $\iint_{x^2+y^2\le r^2} e^{-x^2-y^2}\,dxdy$ (b) $\iint_{x^2+y^2\le 1} \frac{dxdy}{\sqrt{1+x^2+y^2}}$ (c) $\iint_{x^2+y^2\le r^2} x^2 e^{-(x^2+y^2)^2}\,dxdy$

(b)



$0 \le \rho \le 1$

$0 \le \theta \le 2\pi$

$$\iint_{\Omega} \frac{dx\,dy}{\sqrt{1+x^2+y^2}} = \int_{\theta=0}^{\theta=2\pi} \int_{\rho=0}^{\rho=1} \frac{\rho\, d\rho\, d\theta}{\sqrt{1+\rho^2}} =$$

$$= \int_{\theta=0}^{\theta=2\pi} \left(\frac{1}{2} \int_{\rho=0}^{\rho=1} (1+\rho^2)^{-1/2} (2\rho\, d\rho) \right) d\theta =$$

$$\int r^k\, dr = \frac{r^{k+1}}{k+1} + C$$

$$r = 1+\rho^2 \Rightarrow dr = 2\rho\, d\rho$$

$$= \frac{1}{2} \int_{\theta=0}^{\theta=2\pi} \left. \frac{(1+\rho^2)^{1/2}}{\frac{1}{2}} \right|_{\rho=0}^{\rho=1} d\theta =$$

$$= \cancel{\frac{1}{2}} (2\pi - 0) \cdot \cancel{2} \left(\sqrt{1+1^2} - \sqrt{1+0^2} \right)$$

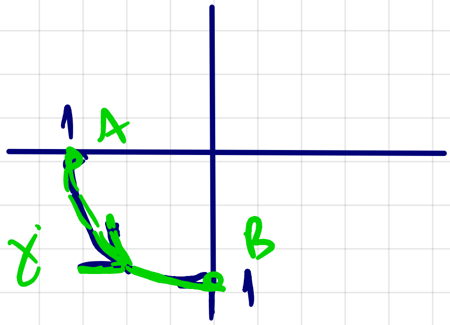
$$= 2\pi \cdot (\sqrt{2} - \sqrt{1}) = 2\pi(\sqrt{2}-1)$$

L 11

1. Calcule a integral de linha de cada campo vetorial $\vec{F}$ dado, ao logo da curva orientada indicada em cada item:

   (a) $\vec{F}(x, y) = (x^2, xy)$, ao longo do segmento de reta de $(0, 0)$ a $(2, 2)$.

   (b) $\vec{F}(x, y) = (4, y)$, no quarto de círculo $x^2 + y^2 = 1$ com $x, y \leq 0$, e orientação anti-horária.



$$\begin{cases} x = \cos t \\ y = \text{sen}\, t. \end{cases} \leftarrow \boxed{\begin{array}{l} x^2 + y^2 = 1 \\ \cos^2 t + \text{sen}^2 t = 1 \end{array}}$$

$$t \in \left[\pi, \frac{3\pi}{2}\right].$$

$$\int_\gamma \vec{F} \cdot d\vec{r} = \int_{\pi}^{\frac{3\pi}{2}} \vec{F}(\gamma(t)) \cdot \gamma'(t)\, dt, \text{ onde}$$

$$\gamma(t) = (\cos t, \text{sen}\, t)$$

$$\Rightarrow \gamma'(t) = (-\text{sen}\, t, \cos t)$$

$$\vec{F}(\gamma(t)) = (4, \text{sen}\, t)$$

Logo:

$$\int_\gamma \vec{F}\, d\vec{r} = \int_{\pi}^{\frac{3\pi}{2}} (4, \text{sen}\, t) \cdot (-\text{sen}\, t, \cos t)\, dt =$$

$$= \int_{\pi}^{\frac{3\pi}{2}} -4\,\text{sen}\, t + \text{sen}\, t \cdot \cos t\, dt =$$

$$= -4 \cdot \int_{\pi}^{\frac{3\pi}{2}} \operatorname{sen} t \, dt + \int_{\pi}^{3\pi} (\operatorname{sen} t)^{1} \underbrace{\cos t \, dt}_{dv} \qquad (\cdots)$$

$$v = \operatorname{sen} t \Rightarrow dv = \cos t \, dt$$

etc.