CÁLCULO 3

$\boxed{12/08/24 — \text{AULA } 39}$

Na aula anterior estudamos a fórmula da mudança de variáveis no $\mathbb{R}^3$. Vejamos exemplos.

01) Calcule $\displaystyle\iiint_\Omega \frac{e^{x-y+z}}{x+y-z}\,dx\,dy\,dz$, onde Ω é a região do $\mathbb{R}^3$ dada por:

$$\begin{cases} 0 \leq x-y+z \leq 1. \\ 1 \leq x+y-z \leq 2 \\ 0 \leq z \leq 1. \end{cases}$$

SOLUÇÃO: Escreva:

$$\begin{cases} u = x-y+z \\ v = x+y-z \\ w = z \end{cases}$$

$+$

$u + v = 2x$

$$\boxed{x = \frac{1}{2}u + \frac{1}{2}v}$$

$$\boxed{z = w}$$

$y = -u + x + z$

$y = -u + \frac{1}{2}u + \frac{1}{2}v + w$

$$\boxed{y = -\frac{1}{2}u + \frac{1}{2}v + w}$$

$$J(T)(u,v,w) = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial y} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\Rightarrow \det J(T)(u,v,w) = \left| \begin{matrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 1 \end{matrix} \right| \begin{matrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{matrix}$$

$$= \frac{1}{4} + 0 + 0 - 0 - 0 + \frac{1}{4} = \frac{1}{2}$$

A região Ω' será:

$$\Omega' : \begin{cases} 0 \le u \le 1 \\ 1 \le v \le 2 \\ 0 \le w \le 1. \end{cases}$$

Assim, teremos:

$$\iiint_{\Omega} \frac{e^{x-y+z}}{x+y-z} \, dx\,dy\,dz = \iiint_{\Omega'} \frac{e^{u}}{v} \underbrace{|\det J(T)(u,v,w)|}_{=\frac{1}{2}} \cdot du\,dv\,dw$$

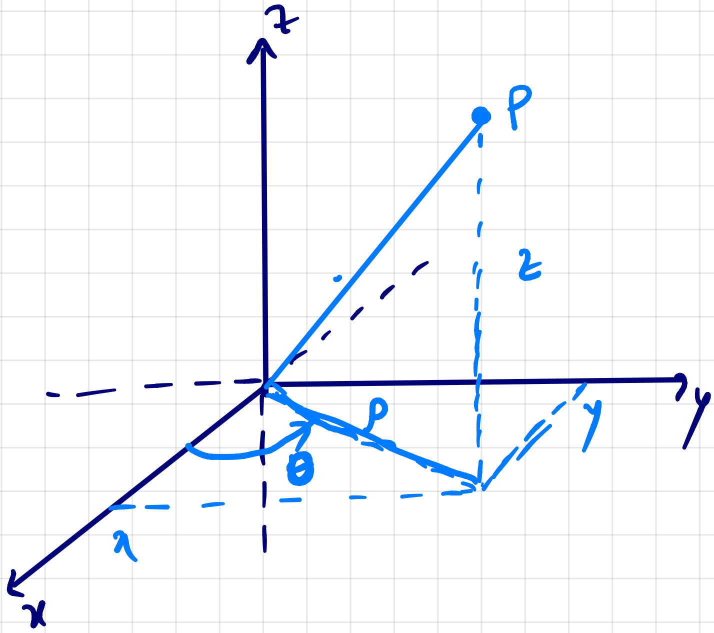
$$= \int_{w=0}^{w=1} \int_{v=1}^{v=2} \int_{u=0}^{u=1} \frac{e^{u}}{v} \cdot \frac{1}{2} \, du\,dv\,dw =$$

$$= \frac{1}{2} \int_{w=0}^{w=1} dw \cdot \int_{v=1}^{v=2} \frac{dv}{v} \cdot \int_{u=0}^{u=1} e^{u}\,du = \frac{1}{2} \left(w \Big|_0^1 \cdot \ln|v| \Big|_1^2 \cdot e^{u} \Big|_0^1 \right) =$$

$$\frac{1}{2}(1-0) \cdot (\ln 2 - \underbrace{\ln 1}_{0}) \cdot (e^1 - e^0) = \frac{1}{2} \ln 2 \cdot (e-1)$$

02) SISTEMA DE COORDENADAS CILÍNDRICAS: Basicamente uma extensão do sist. polar no $\mathbb{R}^3$.

$$P(x,y,z) \longleftrightarrow P(\rho,\theta,z)$$



$$\begin{cases} x = \rho\cos\theta \\ y = \rho\cdot\operatorname{sen}\theta \\ z = z \end{cases}$$

Neste caso, temos

$$J(T)(\rho,\theta,z) = \begin{bmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial y}{\partial \rho} & \frac{\partial z}{\partial \rho} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial z} & \frac{\partial y}{\partial z} & \frac{\partial z}{\partial z} \end{bmatrix}$$

$$\Rightarrow \det J(T)(\rho,\theta,z) = \begin{vmatrix} \cos\theta & \operatorname{sen}\theta & 0 \\ -\rho\operatorname{sen}\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{matrix} \cos\theta & \operatorname{sen}\theta \\ -\rho\operatorname{sen}\theta & \rho\cos\theta \\ 0 & 0 \end{matrix}$$

$$= \rho\cos^2\theta + 0 + 0 - 0 - 0 - (-\rho\operatorname{sen}^2\theta) =$$

$$= \rho\,\underbrace{(\cos^2\theta + \operatorname{sen}^2\theta)}_{=1} = \rho.$$

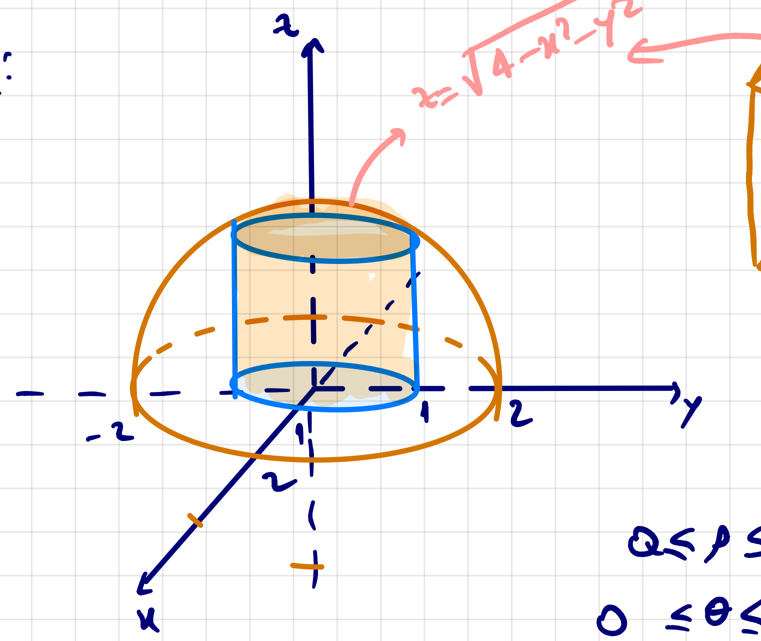
Disso, teremos:

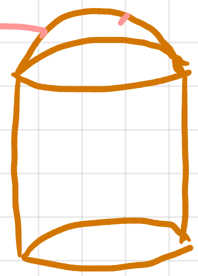
$$\iiint_{\Omega} f(x,y,z)\,dx\,dy\,dz = \iiint_{\Omega'} f(\rho\cos\theta, \rho\,\text{sen}\,\theta, z)\cdot \overbrace{\rho}^{=|\det J(T)(\rho,\theta,z)|}\cdot d\rho\, d\theta\, dz$$

Por exemplo:

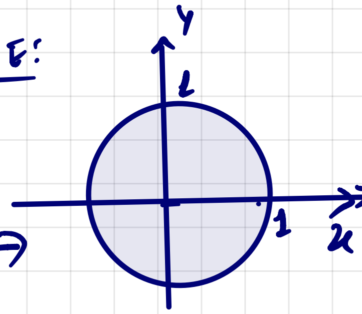
Calcule o volume do sólido abaixo da semi-esfera $x^2+y^2+z^2=4$, $z\geq 0$, limitado pelo cilindro $x^2+y^2=1$ e pelo plano xy, usando integrais triplas.

SOLUÇÃO:





BASE:



$$\left.\begin{array}{l} 0 \leq \rho \leq 1 \\ 0 \leq \theta \leq 2\pi \end{array}\right\}$$

teremos z variando entre:

$$\rho^2 = x^2+y^2$$

$$0 \leq z \leq \sqrt{4-x^2-y^2} = \sqrt{4-\rho^2}$$

Disso, teremos:

$$V = \iiint_{\Omega} dV = \int_{\theta=0}^{\theta=2\pi}\int_{\rho=0}^{\rho=1}\int_{z=0}^{z=\sqrt{4-\rho^2}} \cdot\, \rho\, dz\, d\rho\, d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{\rho=0}^{\rho=1} \rho \cdot z \Big|_{z=0}^{z=\sqrt{4-\rho^2}} d\rho\, d\theta = \int_{\theta=0}^{\theta=2\pi} \int_{\rho=0}^{\rho=1} \left(\sqrt{4-\rho^2} \cdot \rho\, d\rho\right) d\theta$$

$$= \int_0^{2\pi} d\theta \cdot \left(-\frac{1}{2}\right) \int_0^1 (4-\rho^2)^{1/2} \left(\underline{-2\rho\, d\rho}\right) =$$

$$\int u^{\gamma} du \qquad u = 4-\rho^2 \rightarrow du = -2\rho\, d\rho$$

$$= -\frac{1}{2}\ \theta \Big|_0^{2\pi} \frac{(4-\rho^2)^{3/2}}{\frac{3}{2}} \Bigg|_0^1 =$$

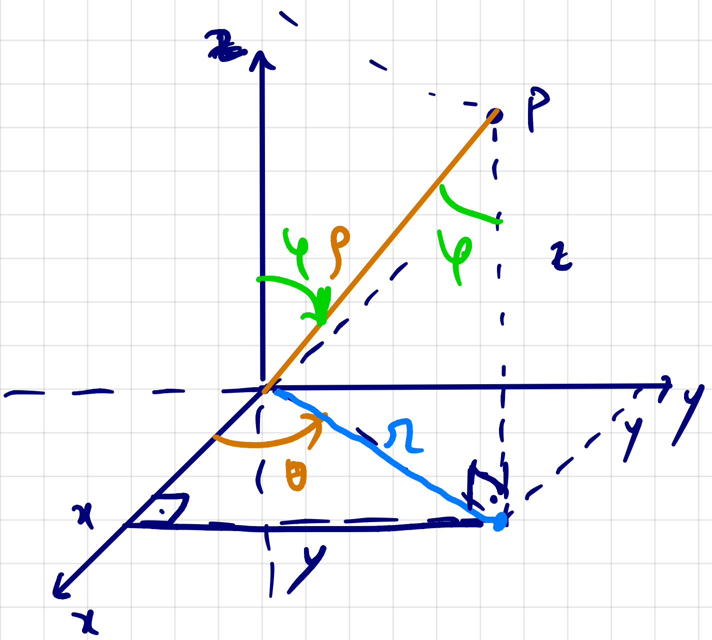
$$- \frac{1}{\cancel{2}} (\cancel{2}\pi) \cdot \frac{2}{3} \cdot \left[(4-1^2)^{3/2} - (4-0^2)^{3/2} \right]$$

$$- \frac{2\pi}{3} \cdot \left(3^{3/2} - 4^{3/2}\right) = \frac{2\pi}{3}\left(4^{3/2} - 3^{3/2}\right)$$

Obs.: O exemplo 03 da aula passada já usou o sist. de coordenadas cilíndricas, sendo naquele caso:

$$\begin{cases} x = x(\rho, \theta) \\ z = z(\rho, \theta) \end{cases} \quad e \quad y = y.$$

SISTEMA DE COORDENADAS ESFÉRICAS:



$P(x,y,z) \leftrightarrow (\rho, \varphi, \theta)$

onde

$0 \leq \varphi \leq \pi$

$0 \leq \theta \leq 2\pi$.

$\rho \in \mathbb{R}$.

Como se faz a mudança do retangular para o esférico?

Note que: $\quad \operatorname{sen}\theta = \dfrac{y}{r} \Rightarrow y = r \cdot \operatorname{sen}\theta$.

$$\cos\theta = \frac{x}{r} \Rightarrow x = r\cos\theta.$$

$$\operatorname{sen}\varphi = \frac{r}{\rho} \Rightarrow \underline{r = \rho\operatorname{sen}\varphi}$$

$$\Rightarrow \quad y = r\operatorname{sen}\theta = \rho\operatorname{sen}\varphi\operatorname{sen}\theta \quad ;$$

$$x = r\cos\theta = \rho\operatorname{sen}\varphi\cos\theta$$

Além disso, temos: $\quad \cos\varphi = \dfrac{z}{\rho} \Rightarrow z = \rho\cos\varphi$

Portanto, temos as relações

$$\begin{cases} x = \rho \operatorname{sen}\varphi \cos\theta \\ y = \rho \operatorname{sen}\varphi \operatorname{sen}\theta \\ z = \rho \cos\varphi \end{cases}$$

$$\det J(T)(\rho,\varphi,\theta) = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial y}{\partial \rho} & \frac{\partial z}{\partial \rho} \\ \frac{\partial x}{\partial \varphi} & \frac{\partial y}{\partial \varphi} & \frac{\partial z}{\partial \varphi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \end{vmatrix} =$$

$$= \begin{vmatrix} \operatorname{sen}\varphi\cos\theta & \operatorname{sen}\varphi\operatorname{sen}\theta & \cos\varphi \\ \rho\cos\varphi\cos\theta & \rho\cos\varphi\operatorname{sen}\theta & -\rho\operatorname{sen}\varphi \\ -\rho\operatorname{sen}\varphi\operatorname{sen}\theta & \rho\operatorname{sen}\varphi\cos\theta & 0 \end{vmatrix} \begin{matrix} \operatorname{sen}\varphi\cos\theta & \operatorname{sen}\varphi\operatorname{sen}\theta \\ \rho\cos\varphi\cos\theta & \rho\cos\varphi\operatorname{sen}\theta \\ -\rho\operatorname{sen}\varphi\operatorname{sen}\theta & \rho\operatorname{sen}\varphi\cos\theta \end{matrix}$$

$$\begin{matrix} - & - & - & + & + & + \end{matrix}$$

$$= 0 + \rho^2 \operatorname{sen}^3\varphi \operatorname{sen}^2\theta + \rho^2\cos^2\varphi . \operatorname{sen}\varphi . \cos^2\theta +$$

$$+ \rho^2 \operatorname{sen}\varphi \cos^2\varphi \operatorname{sen}^2\theta + \rho^2 \operatorname{sen}^3\varphi \cos^2\theta + 0$$

$$= \rho^2 \operatorname{sen}\varphi \left(\underline{\operatorname{sen}^2\varphi \operatorname{sen}^2\theta} + \underline{\cos^2\varphi\cos^2\theta} + \underline{\cos^2\varphi\operatorname{sen}^2\theta} + \underline{\operatorname{sen}^2\varphi\cos^2\theta} \right)$$

$$= \rho^2 \operatorname{sen}\varphi . \left(\operatorname{sen}^2\theta \underbrace{(\operatorname{sen}^2\varphi + \cos^2\varphi)}_{=1} + \cos^2\theta \underbrace{(\cos^2\varphi + \operatorname{sen}^2\varphi)}_{=1} \right)$$

$$= \rho^2 \operatorname{sen}\varphi \, (\underbrace{\operatorname{sen}^2\theta + \cos^2\theta}_{=1}) = \underline{\rho^2 \operatorname{sen}\varphi.}$$

Note também que

$$x^2+y^2+z^2 = \rho^2 \operatorname{sen}^2\varphi \cos^2\theta + \rho^2 \operatorname{sen}^2\varphi \operatorname{sen}^2\theta + \rho^2\cos^2\varphi$$

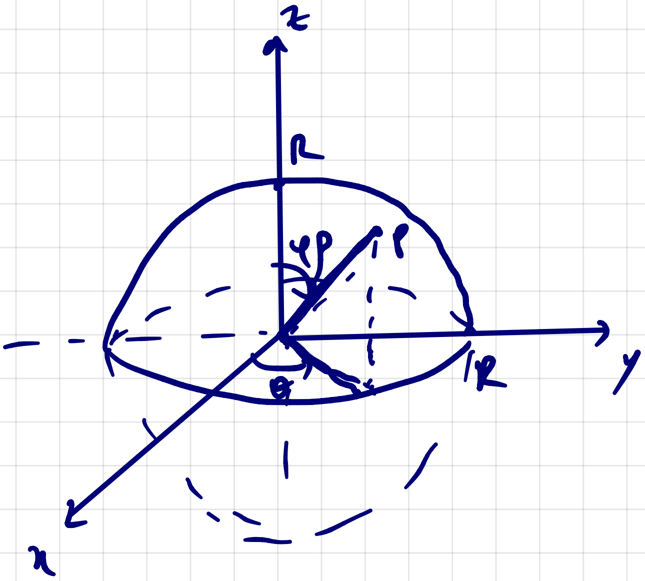
$$= \rho^2 \left(\underbrace{\operatorname{sen}^2\varphi \underbrace{(\cos^2\theta + \operatorname{sen}^2\theta)}_{=1} + \cos^2\varphi}_{} \right) = \rho^2$$

Assim, dada $f: \Omega \subset \mathbb{R}^3 \to \mathbb{R}$ integrável, no sistema esférico, teremos:

$$\iiint_{\Omega} f(x,y,z)\,dx\,dy\,dz = \iiint_{\Omega'} f(x(\rho,\varphi,\theta), y(\rho,\varphi,\theta), z(\rho,\varphi,\theta)) \cdot \rho^2 \operatorname{sen}\varphi \, d\rho\, d\varphi\, d\theta$$

Ex-1(a) Deduzir a fórmula do volume de uma esfera de raio R, usando o sist. de coordenadas esféricas.

Solução:



$0 \leq \theta \leq 2\pi$

$0 \leq \rho \leq R$

$0 \leq \varphi \leq \pi$

$$V = \iiint_{\Omega} dV = \int_{\theta=0}^{\theta=2\pi} \int_{\varphi=0}^{\varphi=\pi} \int_{\rho=0}^{\rho=R} \rho^2 \operatorname{sen}\varphi \cdot d\rho\, d\varphi\, d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{\varphi=0}^{\varphi=\pi} \operatorname{sen}\varphi \int_{\rho=0}^{\rho=R} \rho^2 d\rho \; d\varphi \; d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} d\theta \cdot \int_{\varphi=0}^{\varphi=\pi} \operatorname{sen}\varphi \, d\varphi \cdot \int_{\rho=0}^{\rho=R} \rho^2 d\rho =$$

$$= \theta \Big|_{\theta=0}^{\theta=2\pi} \cdot (-\cos\varphi)\Big|_{\varphi=0}^{\varphi=\pi} \left(\frac{\rho^3}{3}\right)\Big|_{\rho=0}^{\rho=R} =$$

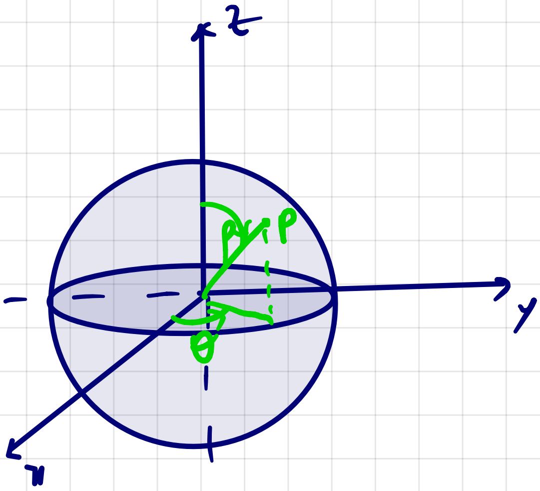
$$= 2\pi \cdot \underbrace{(-\cos\pi + \cos 0)}_{2} \cdot \left(\frac{R^3}{3} - 0\right) = 4\pi \cdot \frac{R^3}{3} = \frac{4\pi R^3}{3}$$

(b) Calcule $\iiint_{\Omega} e^{(x^2+y^2+z^2)^{\frac{3}{2}}} dV$, sendo

$$\Omega = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \leq 1\}.$$

Solução:

Como $x^2+y^2+z^2=\rho^2$,



$$\left.\begin{array}{l} 0 \leq \varphi \leq \pi \\ 0 \leq \theta \leq 2\pi \\ 0 \leq \rho \leq 1. \end{array}\right\} \Omega$$

teremos:

$$\iiint_{\Omega} e^{(x^2+y^2+z^2)^{3/2}}\, dV = \int_{\theta=0}^{\theta=2\pi}\int_{\varphi=0}^{\varphi=\pi}\int_{\rho=0}^{\rho=1} e^{(\rho^{\cancel{2}})^{3/\cancel{2}}}\rho^2 \operatorname{sen}\varphi \, d\rho\, d\varphi\, d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi}\int_{\varphi=0}^{\varphi=\pi} \operatorname{sen}\varphi \left(\frac{1}{3}\int_{\rho=0}^{\rho=1} e^{\rho^3}\left(3\rho^2 d\rho\right)\right) d\varphi\, d\theta =$$

$\int e^r dr$

$r = \rho^3$

$\Rightarrow dr = 3\rho^2 d\rho$

$$= \frac{1}{3} \cdot \int_{\theta=0}^{\theta=2\pi} d\theta \cdot \int_{\varphi=0}^{\varphi=\pi} \operatorname{sen}\varphi \, d\varphi \cdot e^{\rho^3}\Big|_{\rho=0}^{\rho=1} =$$

$$= \frac{1}{3}\, \theta\Big|_0^{2\pi} \cdot (-\cos\varphi)\Big|_0^{\pi} \cdot (e^1 - e^0)$$

$$\frac{2\pi}{3} \cdot (-\cos\pi + \cos 0) \cdot (e-1) = \frac{4\pi}{3}(e-1)$$

(c) Use coordenadas esféricas para determinar o volume do sólido acima do cone $z = \sqrt{x^2+y^2}$ e abaixo da esfera $x^2+y^2+z^2 = z$

(EXERCÍCIO)