CÁLCULO 3

22/07/24 - AULA 29

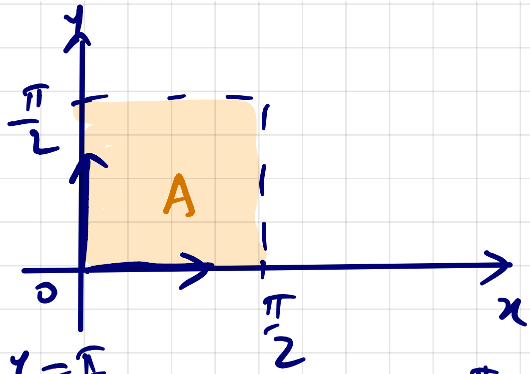
Vejamos mais um exemplo do CASO 1:

L8

1. Calcule cada integral dupla a seguir:

(a) $\iint_A \operatorname{sen}(x+y)dxdy$, onde $A = [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$.

Solução:



$$\int_{y=0}^{y=\frac{\pi}{2}} \left(\int_{x=0}^{x=\frac{\pi}{2}} \operatorname{sen}(x+y)\, dx \right) dy = \int_{y=0}^{y=\frac{\pi}{2}} \left[-\cos(x+y) \Big|_{x=0}^{x=\frac{\pi}{2}} \right] dy$$

$\int \operatorname{sen} u\, du = -\cos u + C$

$u = x+y \Rightarrow du = dx$

$$= \int_{y=0}^{y=\frac{\pi}{2}} \left[-\cos\left(\frac{\pi}{2}+y\right) + \cos(0+y) \right] dy =$$

$$= -\int_0^{\frac{\pi}{2}} \cos\left(\frac{\pi}{2}+y\right) dy + \int_0^{\frac{\pi}{2}} \cos y\, dy =$$

$\int \cos u\, du = \operatorname{sen} u + C$

$$= -\operatorname{sen}\left(\frac{\pi}{2}+y\right)\Big|_0^{\frac{\pi}{2}} + \operatorname{sen} y\Big|_0^{\frac{\pi}{2}} =$$

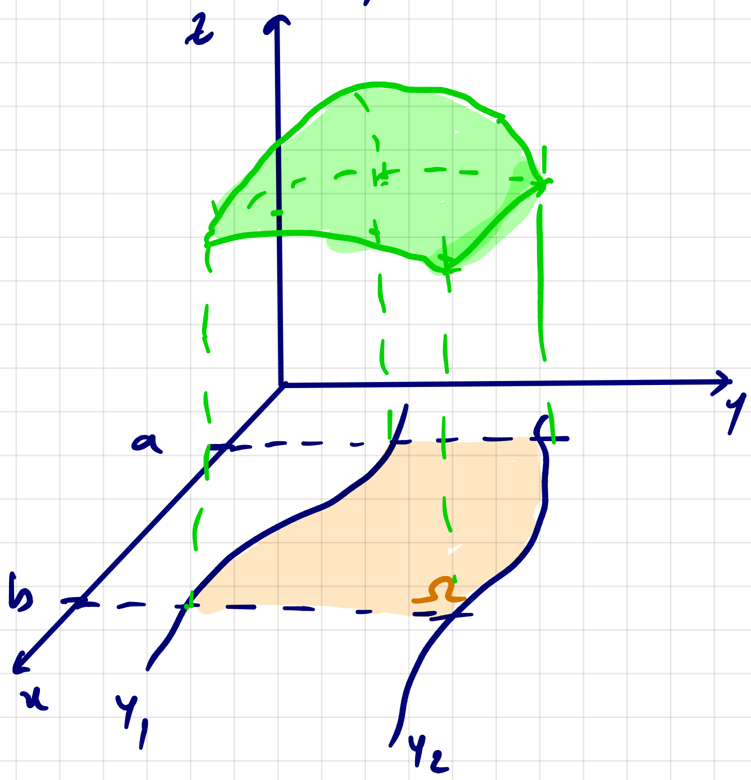
$$= -\operatorname{sen}\left(\frac{\pi}{2}+\frac{\pi}{2}\right) + \operatorname{sen}\left(\frac{\pi}{2}\right) + \operatorname{sen}\left(\frac{\pi}{2}\right) - \operatorname{sen} 0$$

$$= 0 + 1 + 1 - 0 = 2$$

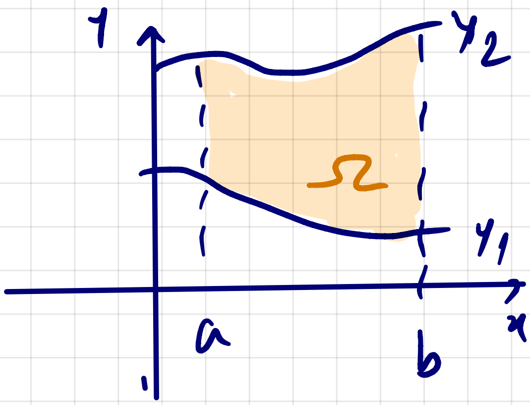
Vejamos o 2º CASO:

2º CASO (CASO GERAL) QUANDO A REGIÃO Ω DEPENDE DE, PELO MENOS, UMA CURVA.

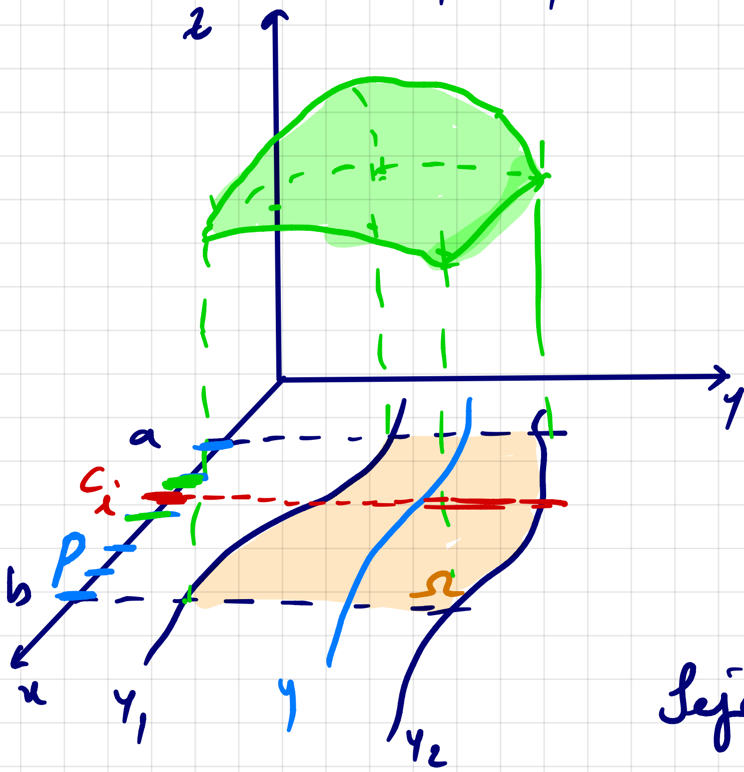
Para dar um significado geométrico de volume, assuma $f \geq 0$, onde $f: \Omega \subset \mathbb{R}^2 \to \mathbb{R}$, e Ω uma região limitada por duas curvas y_1 e y_2, c.f. o esquema abaixo; no intervalo $[a, b]$ no eixo ox.

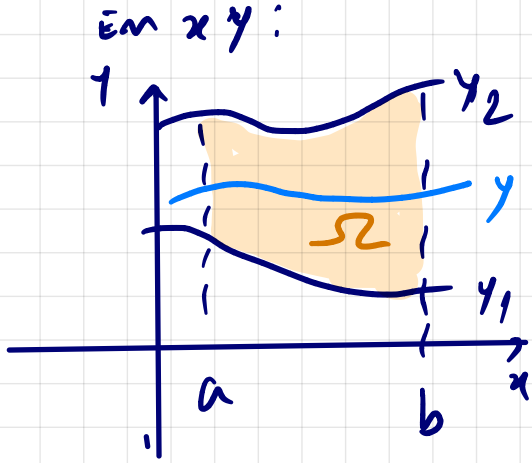


Em xy:



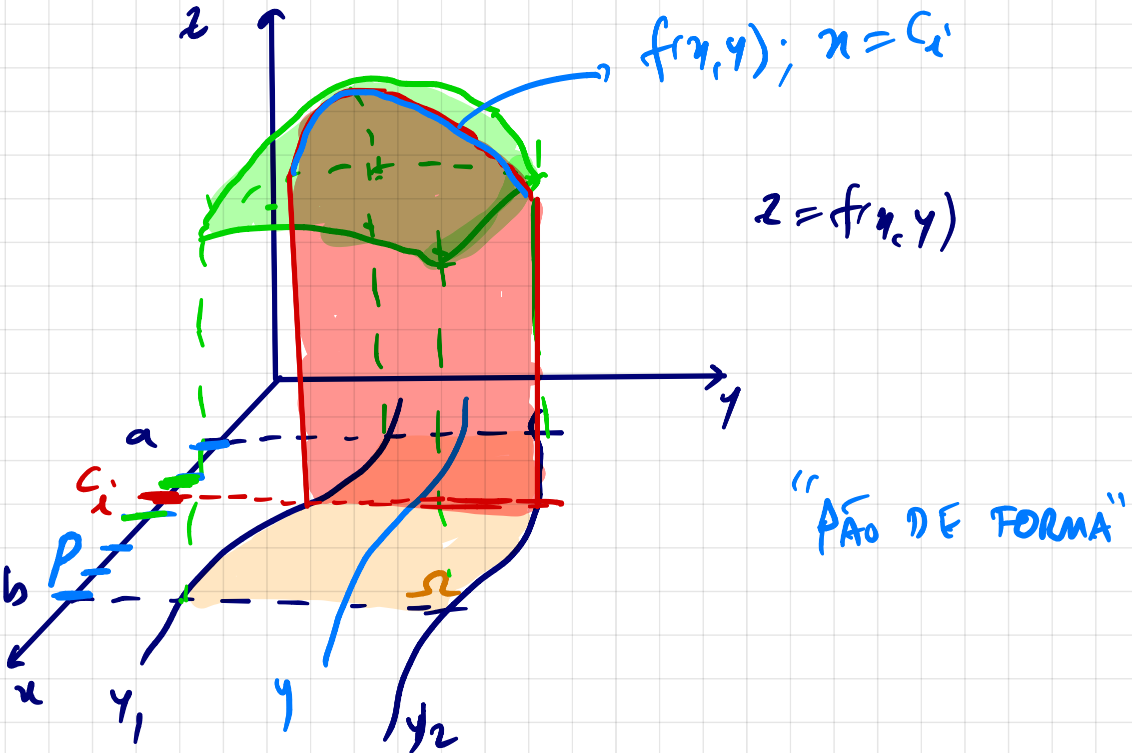
Seja y uma curva qualquer entre y_1 e y_2; e seja $P = \{a = t_0 < t_1 < \dots < t_n = b\}$ uma partição do intervalo $[a,b]$, no eixo ox.





Seja $c_i \in [t_{i-1}, t_i]$

Considere a lâmina formada pelo plano paralelo ao plano yz, passando pelo corte c_i no plano xy; abaixo do gráfico da superfície dada por f.



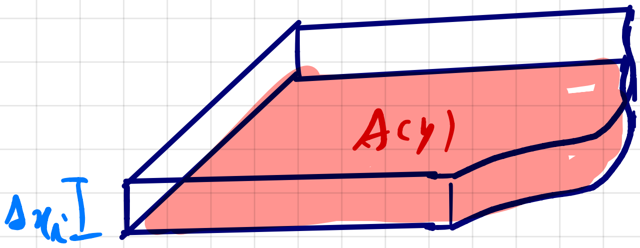
A área $A(y)$, destacada em vermelho, formada pela lâmina, é dada por:

$$A(y) = \int_{y_1}^{y_2} f(c_i, y)\, dy$$

Sendo $\Delta x_i = t_i - t_{i-1}$ a "espessura" de uma lâmina, o volume aproximado $\tilde{V}$ do sólido abaixo do gráfico de f em Ω, será:

$$\tilde{V} = \sum_{i=1}^{n} A(y) \cdot \Delta x_i$$

(PRINCÍPIO DE CAVALIERI)



FATIA DO "PÃO DE FORMA".

O volume V será:

$$V = \lim_{n \to \infty} \tilde{V} = \lim_{n \to \infty} \sum_{i=1}^{n} A(y) \cdot \Delta x_i$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} \int_{y_1}^{y_2} f(c_i, y)\, dy \cdot \Delta x_i$$

$$= \int_a^b \left(\int_{y_1}^{y_2} f(x, y)\, dy \right) dx.$$

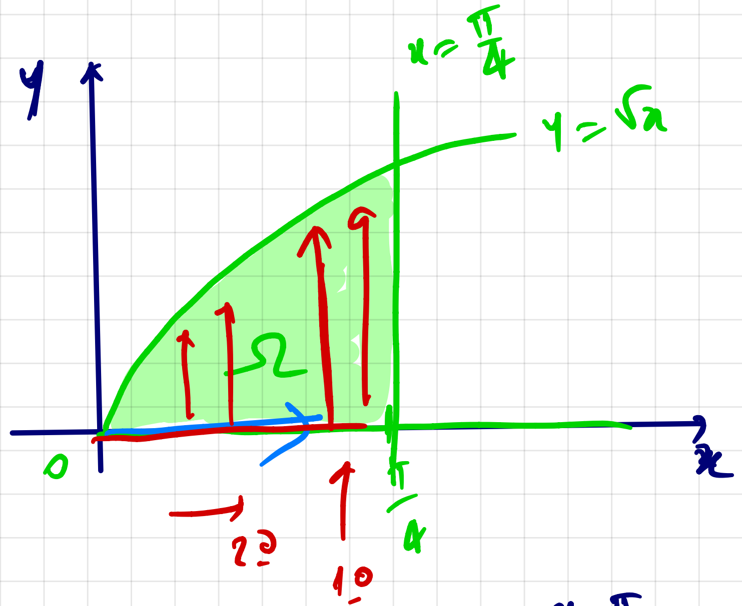
Poderíamos calcular também, primeiro em x, e depois em y, dependendo do problema.
A única exigência é que a integral mais externa fique com limites de integração constantes.

<u>Vejamos exemplos</u>:

01) Calcule $\displaystyle\iint_{\Omega} \sqrt{x}\cdot\cos(y\sqrt{x})\,dx\,dy$, onde Ω é formada pelas retas $y=0$; $x=\frac{\pi}{4}$ e pela curva $y=\sqrt{x}$.

$\underbrace{y=0}_{\text{eixo } x}$

<u>Solução</u>:



$$\iint_{\Omega} \sqrt{x}\cdot\cos(y\sqrt{x})\,dx\,dy = \int_{x=0}^{x=\frac{\pi}{4}}\left(\int_{y=0}^{y=\sqrt{x}} \underbrace{\sqrt{x}}_{\text{CONSTANTE PARA } y}\cos(y\sqrt{x})\,dy\right)dx$$

$$= \int_{x=0}^{x=\frac{\pi}{4}} \sqrt{x}\left(\int_{y=0}^{y=\sqrt{x}} \cos(y\sqrt{x})\,dy\right)dx =$$

$\int \cos v\,dv = \operatorname{sen} v + C$

$$u = y\sqrt{x} \Rightarrow du = \underbrace{\sqrt{x}\,dy}$$

$$= \int_{x=0}^{x=\frac{\pi}{4}} \left[\int_{y=0}^{y=\sqrt{x}} \cos(\underbrace{y\sqrt{x}}_{u}) \, (\underbrace{\sqrt{x}\,dy}_{du}) \right] dx$$

$$= \int_{x=0}^{x=\frac{\pi}{4}} \operatorname{sen}(y\sqrt{x}) \Big|_{y=0}^{y=\sqrt{x}} dx = \int_{x=0}^{x=\frac{\pi}{4}} (\operatorname{sen} x - \underbrace{\operatorname{sen} 0}_{0}) \, dx$$

$$= \int_0^{\frac{\pi}{4}} \operatorname{sen} x \, dx = -\cos x \Big|_0^{\frac{\pi}{4}} = -\left(\cos\frac{\pi}{4} - \cos 0 \right)$$

$$= -\left(\frac{\sqrt{2}}{2} - 1 \right) = -\frac{\sqrt{2}}{2} + 1 = \frac{2-\sqrt{2}}{2}$$

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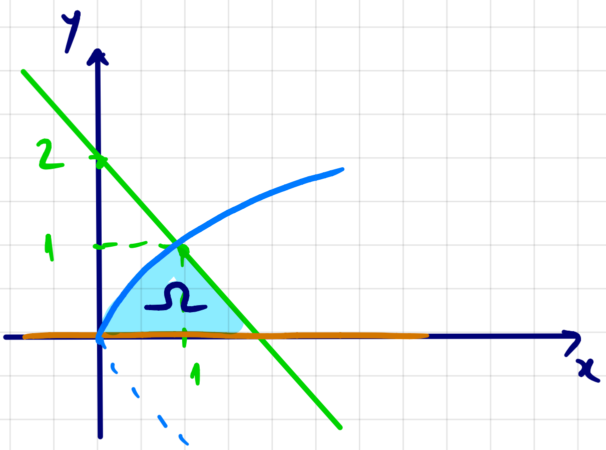
02) Calcule $\iint_\Omega x\sqrt{y}\,dx\,dy$, onde Ω é a região formada pelas retas $y=0$; $x+y=2$ e pela parábola $x=y^2$, no 1º q.

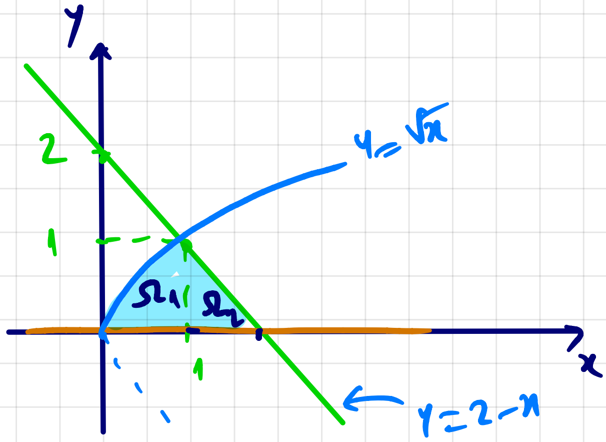
eixo x

$y = 2-x$

SOLUÇÃO:

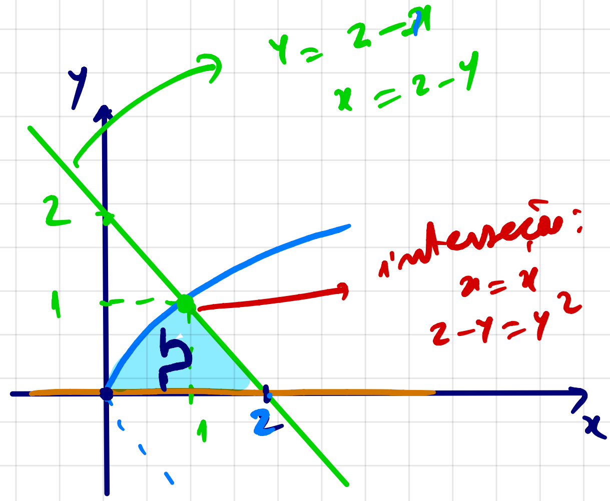


A priori, podemos calcular $\iint_\Omega f$ de duas formas:



$$\iint_\Omega f = \iint_{\Omega_1} f + \iint_{\Omega_2} f$$

$$= \int_{x=0}^{x=1}\int_{y=0}^{y=\sqrt{x}} f + \int_{x=1}^{x=2}\int_{y=0}^{y=2-x} f$$



$$\iint_\Omega f = \int_{y=0}^{y=1}\int_{x=y^2}^{x=2-y} f$$

este caso é mais vantajoso, pois consiste em um cálculo de uma integral dupla, apenas.

$$\int_{y=0}^{y=1}\left(\int_{x=y^2}^{x=2-y} x\sqrt{y}\,dx\right)dy = \int_{y=0}^{y=1}\sqrt{y}\cdot\left(\int_{x=y^2}^{x=2-y} x\,dx\right)dy =$$

$$= \int_{y=0}^{y=1} \left.\frac{x^2}{2}\right|_{x=y^2}^{x=2-y} dy = \int_{y=0}^{y=1}\sqrt{y}\left(\frac{(2-y)^2}{2} - \frac{y^4}{2}\right)dy =$$

$$= \int_0^1 \sqrt{y}\cdot\left(\frac{4-2y+y^2}{2} - \frac{y^4}{2}\right)dy = \frac{1}{2}\int_0^1 \left(4\sqrt{y} - 2y\sqrt{y} + y^2\sqrt{y} - y^4\sqrt{y}\right)dy$$

$$= \frac{1}{2}\int_0^1 \left(4y^{\frac{1}{2}} - 2y\cdot y^{\frac{1}{2}} + y^2\cdot y^{\frac{1}{2}} - y^4\cdot y^{\frac{1}{2}}\right)dy$$

$$= \frac{4}{2}\cdot\int_0^1 y^{\frac{1}{2}}dy - \frac{\cancel{2}}{\cancel{2}}\int_0^1 y^{\frac{3}{2}}dy + \frac{1}{2}\int_0^1 y^{\frac{5}{2}}dy - \frac{1}{2}\int y^{\frac{9}{2}}dy$$

$$= 2\cdot\left.\frac{y^{3/2}}{\frac{3}{2}}\right|_0^1 - \left.\frac{y^{5/2}}{\frac{5}{2}}\right|_0^1 + \frac{1}{2}\cdot\left.\frac{y^{\frac{7}{2}}}{\frac{7}{2}}\right|_0^1 - \frac{1}{2}\cdot\left.\frac{y^{\frac{11}{2}}}{\frac{11}{2}}\right|_0^1 =$$

$$2\cdot\frac{2}{3}\left.y^{3/2}\right|_0^1 - \frac{2}{5}\left.y^{5/2}\right|_0^1 + \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{7}\cdot\left.y^{\frac{7}{2}}\right|_0^1 - \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{11}\cdot\left.y^{\frac{11}{2}}\right|_0^1$$

$$= \frac{4}{3}\cdot 1 - \frac{2}{5}\cdot 1 + \frac{1}{7}\cdot 1 - \frac{1}{11}\cdot 1 =$$

$$= \frac{4}{3} - \frac{2}{5} + \frac{1}{7} - \frac{1}{11} = \ldots$$

03) Obtenha o volume do tetraedro limitado pelos planos $x+2y+z=2$, $x=2y$, $x=0$ e $z=0$

(EXERCÍCIO) $z=f(x,y)$

(Resp: $\frac{1}{3}$)

RESOLUÇÃO DE EXERCÍCIOS (PREPARO P/ PROVA 2)

L5:

7. Se $u = f(x, y)$ é uma função diferenciável de x e y com $x = \rho\cos\theta$ e $y = \rho\sin\theta$, mostre que

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \rho}\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{\rho}$$

e

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \rho}\sin\theta + \frac{\partial u}{\partial \theta}\frac{\cos\theta}{\rho}$$

Solução: Note que:

$$u = u(x, y)$$

$$\begin{tikzcd}[column sep=small]
 & & & u \arrow[dl] \arrow[dr] & & & \\
 & & x \arrow[dl] \arrow[d] & & y \arrow[d] \arrow[dr] & & \\
 & \rho & \theta & & \rho & \theta &
\end{tikzcd}$$

Pela Regra da Cadeia:

$$\frac{\partial u}{\partial \rho} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \rho} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \rho}$$

$$x = \rho\cos\theta \quad \frac{\partial x}{\partial \rho} = \cos\theta, \quad \frac{\partial x}{\partial \theta} = -\rho\,\mathrm{sen}\,\theta$$

$$y = \rho\,\mathrm{sen}\,\theta \quad \frac{\partial y}{\partial \rho} = \mathrm{sen}\,\theta, \quad \frac{\partial y}{\partial \theta} = \rho\cos\theta$$

$$\boxed{\frac{\partial u}{\partial \rho} = \frac{\partial u}{\partial x}\cdot\cos\theta + \frac{\partial u}{\partial y}\cdot\mathrm{sen}\,\theta} \quad (I)$$

Também, pela R. da Cadeia, vem:

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

$$\boxed{\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}(-\rho\,\mathrm{sen}\,\theta) + \frac{\partial u}{\partial y}\cdot\rho\cos\theta} \quad (II)$$

A ideia agora é trabalhar com (I) e (II) procurando isolar $\frac{\partial u}{\partial x}$ e $\frac{\partial u}{\partial y}$ em função das outras.

$$\begin{cases} \frac{\partial u}{\partial \rho} = \frac{\partial u}{\partial x} \cdot \cos\theta + \frac{\partial u}{\partial y} \cdot \operatorname{sen}\theta & (\times \rho \operatorname{sen}\theta) \\ \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}(-\rho \operatorname{sen}\theta) + \frac{\partial u}{\partial y} \cdot \rho\cos\theta & (\times \cos\theta) \end{cases}$$

$$\rho \cdot \operatorname{sen}\theta \cdot \frac{\partial u}{\partial \rho} = \cancel{\frac{\partial u}{\partial x} \cdot \rho \operatorname{sen}\theta \cdot \cos\theta} + \frac{\partial u}{\partial y} \cdot \rho \operatorname{sen}^2\theta$$

$$+ \quad \cos\theta \cdot \frac{\partial u}{\partial \theta} = \cancel{-\frac{\partial u}{\partial x} \cdot \rho \operatorname{sen}\theta \cdot \cos\theta} + \frac{\partial u}{\partial y} \cdot \rho\cos^2\theta$$

$$\rho \operatorname{sen}\theta \cdot \frac{\partial u}{\partial \rho} + \cos\theta \cdot \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial y} \rho \cdot \underbrace{(\operatorname{sen}^2\theta + \cos^2\theta)}_{=1}$$

$$\frac{\partial u}{\partial y} = \frac{\cancel{\rho} \cdot \operatorname{sen}\theta}{\cancel{\rho}} \frac{\partial u}{\partial \rho} + \frac{\cos\theta}{\rho} \cdot \frac{\partial u}{\partial \theta}$$

$$\Rightarrow \boxed{\frac{\partial u}{\partial y} = \operatorname{sen}\theta \cdot \frac{\partial u}{\partial \rho} + \frac{1}{\rho} \cdot \cos\theta \cdot \frac{\partial u}{\partial \theta}}$$

Assim, de (I), vem:

$$\frac{\partial u}{\partial \rho} = \frac{\partial u}{\partial x} \cdot \cos\theta + \frac{\partial u}{\partial y} \cdot \operatorname{sen}\theta$$

$$\frac{\partial u}{\partial \rho} = \frac{\partial u}{\partial x} \cdot \cos\theta + \left(\operatorname{sen}\theta \cdot \frac{\partial u}{\partial \rho} + \frac{1}{\rho} \cos\theta \, \frac{\partial u}{\partial \theta}\right) \cdot \operatorname{sen}\theta$$

$$\frac{\partial u}{\partial \rho} = \frac{\partial u}{\partial x}\cdot\cos\theta + \operatorname{sen}^2\theta\cdot\frac{\partial u}{\partial \rho} + \frac{1}{\rho}\operatorname{sen}\theta\cos\theta\cdot\frac{\partial u}{\partial \theta}$$

$$\frac{\partial u}{\partial x}\cdot\cos\theta = \frac{\partial u}{\partial \rho} - \operatorname{sen}^2\theta\,\frac{\partial u}{\partial \rho} - \frac{1}{\rho}\operatorname{sen}\theta\cdot\cos\theta\,\frac{\partial u}{\partial \theta}$$

$$\frac{\partial u}{\partial x}\cdot\cos\theta = \frac{\partial u}{\partial \rho}\cdot\big(\overbrace{1-\operatorname{sen}^2\theta}^{\cos^2\theta}\big) - \frac{1}{\rho}\operatorname{sen}\theta\cos\theta\cdot\frac{\partial u}{\partial \theta} \quad \div\cos\theta$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \rho}\cdot\frac{\cos^{\cancel{2}}\theta}{\cancel{\cos\theta}} - \frac{1}{\rho}\,\frac{\operatorname{sen}\theta\cdot\cancel{\cos\theta}}{\cancel{\cos\theta}}\cdot\frac{\partial u}{\partial \theta}$$

$$\boxed{\frac{\partial u}{\partial x} = \cos\theta\cdot\frac{\partial u}{\partial \rho} - \frac{1}{\rho}\operatorname{sen}\theta\cdot\frac{\partial u}{\partial \theta}}$$