CÁLCULO 3

10/07/24 - AULA 24

Na aula anterior estudamos a FÓRMULA DE TAYLOR com o resto na forma de Lagrange, para funções de $\mathbb{R}^2$ em $\mathbb{R}$.

O objetivo é aproximar a f por uma função polinomial a duas variáveis reais.

Vimos que, sendo $x = x_0 + h$ e $y = y_0 + k$, onde $h, k \in \mathbb{R}$ são incrementos, então

$$f(x_0+h, y_0+k) = f(x_0, y_0) + \left(\frac{\partial}{\partial x} \cdot h + \frac{\partial}{\partial y} k\right) f(x_0, y_0) +$$

$$+ \frac{1}{2!} \left(\frac{\partial}{\partial x} h + \frac{\partial}{\partial y} k\right)^{(2)} f(x_0, y_0) +$$

$$+ \frac{1}{3!} \left(\frac{\partial}{\partial x} h + \frac{\partial}{\partial y} k\right)^{(3)} f(x_0, y_0) +$$

$$\vdots$$

$$+ \frac{1}{n!} \left(\frac{\partial}{\partial x} h + \frac{\partial}{\partial y} k\right)^{(n)} \cdot f(x_0, y_0) + R_n \quad , \text{ onde}$$

$$R_n = \frac{1}{(n+1)!} \cdot \left(\frac{\partial}{\partial x} h + \frac{\partial}{\partial y} k\right)^{(n+1)} \cdot f(\theta h, \theta k) ; \quad 0 < \theta < 1.$$

Lembrando que, usamos a notação:

$$\left(\frac{\partial}{\partial x}h + \frac{\partial}{\partial y}k\right)^{(2)} f(x_0,y_0) =$$

$$\frac{\partial^2}{\partial x^2} f(x_0,y_0)\cdot h^2 + 2\cdot\frac{\partial^2}{\partial x\partial y} f(x_0,y_0)\cdot hk + \frac{\partial^2}{\partial y^2} f(x_0,y_0),$$

et cetera.

Resolvendo o exemplo deixado no final da aula anterior:

<u>EXEMPLO.</u> Expandir $f(x,y) = \operatorname{sen} xy$ em potências de x e de y no ponto $P(0, \frac{\pi}{2})$.

<u>SOLUÇÃO:</u> Sejam $h, k \in \mathbb{R}$ incrementos tais que

$$x = x_0 + h \quad \text{e} \quad y = y_0 + k.$$

$$f(x,y) = \operatorname{sen} xy$$

$$\frac{\partial f}{\partial x} = y\cdot\cos xy$$

$$\frac{\partial^2 f}{\partial x^2} = -y^2\cdot\operatorname{sen} xy$$

$$\frac{\partial^3 f}{\partial x^3} = -y^3\cos xy$$

$$\frac{\partial^3 f}{\partial x^2\partial y} = -y^2 x\cdot\cos xy - 2y\operatorname{sen} xy$$

$$\dots$$

$$\frac{\partial^2 f}{\partial x\partial y} = -xy\operatorname{sen} xy$$

$$\frac{\partial^3 f}{\partial x\partial y^2} = -x^2 y\cdot\cos xy - x\operatorname{sen} xy$$

$$\frac{\partial f}{\partial y} = x\cdot\cos xy$$

$$\frac{\partial^2 f}{\partial y^2} = -x^2\cdot\operatorname{sen} xy$$

$$\frac{\partial^3 f}{\partial y^3} = -x^3\cos xy$$

$$\dots$$

Aplicando em $P\left(\underset{x_0}{0}, \underset{y_0}{\frac{\pi}{2}}\right)$, temos:

$f(x_0, y_0) = \operatorname{sen} 0 = 0$

- $\underbrace{\frac{\partial f}{\partial x}(x_0, y_0) = \frac{\pi}{2} \cdot \cos 0 = \frac{\pi}{2}}$; $\underbrace{\frac{\partial f}{\partial y}(x_0, y_0) = 0}$

- $\frac{\partial^2 f}{\partial x^2}(x_0, y_0) = -\left(\frac{\pi}{2}\right)^2 \cdot \operatorname{sen} 0 = 0$; $\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) = 0$; $\underbrace{\frac{\partial^2 f}{\partial y^2}(x_0, y_0) = 0}_{=0}$

- $\frac{\partial^3 f}{\partial x^3}(x_0, y_0) = -\left(\frac{\pi}{2}\right)^3 \cdot \cos 0 = -\frac{\pi^3}{8}$; $\frac{\partial^3 f}{\partial x^2 \partial y}(x_0, y_0) = 0$; $\underbrace{\frac{\partial^3 f}{\partial y^3}(x_0, y_0) = 0}_{=0}$

Assim, temos

$$f(\overbrace{x_0 + h}^{x}, \overbrace{y_0 + k}^{y}) = f(x_0, y_0) + \left(\frac{\partial}{\partial x} h + \frac{\partial}{\partial y} k\right) f(x_0, y_0) +$$

$$+ \frac{1}{2!}\left(\frac{\partial}{\partial x} h + \frac{\partial}{\partial y} k\right)^{(2)} f(x_0, y_0) +$$

$$+ \frac{1}{3!}\left(\frac{\partial}{\partial x} h + \frac{\partial}{\partial y} k\right)^{(3)} f(x_0, y_0) + R_3$$

$$f(x, y) = 0 + \overbrace{\frac{\partial}{\partial x} f(x_0, y_0)}^{=\frac{\pi}{2}} \cdot h + \overbrace{\frac{\partial}{\partial y} f(x_0, y_0)}^{=0} \cdot k +$$

$$+ \frac{1}{2!}\left(\underbrace{\frac{\partial^2}{\partial x^2} f(x_0, y_0)}_{=0} h^2 + 2 \cdot \underbrace{\frac{\partial^2}{\partial x \partial y} f(x_0, y_0)}_{=0} \cdot hk + \underbrace{\frac{\partial^2 f}{\partial y^2}(x_0, y_0) \cdot k^2}_{0}\right) +$$

$$+\frac{1}{3!}\left(\underbrace{\frac{\partial^3}{\partial x^3}f(x_0,y_0)}_{-\frac{\pi^3}{8}}\cdot h^3+3\cdot\underbrace{\frac{\partial^2 f}{\partial x^2}(x_0,y_0)}_{=0}\cdot\underbrace{\frac{\partial f}{\partial y}(x_0,y_0)}_{=0}h^2k\ +\right.$$

$$\left.+3\cdot\underbrace{\frac{\partial f}{\partial x}(x_0,y_0)}_{\frac{\pi}{2}}\cdot\underbrace{\frac{\partial^2 f}{\partial y^2}(x_0,y_0)}_{=0}h\,k^2+\underbrace{\frac{\partial^3}{\partial y^3}f(x_0,y_0)}_{=0}k^3\right)+R_3$$

$$\Rightarrow f(x,y)=\frac{\pi}{2}\cdot h+\frac{1}{3!}\left(-\frac{\pi^3}{8}\cdot h^3\right)+R_3\ ;$$

e como $x=x_0+h$ e $y=y_0+k$

$$\Rightarrow h=x-x_0\ ;\quad k=y-y_0$$

$$\Rightarrow h=x-0=x\ ;\quad k=y-\frac{\pi}{2}$$

Então, obtemos:

$$f(x,y)=\frac{\pi}{2}\cdot x+\frac{1}{3!}\left(-\frac{\pi^3}{8}\cdot x^3\right)+R_3$$

$$f(x,y)=\frac{\pi}{2}x-\frac{\pi^3}{24}x^3+R_3\ .$$

Ou seja, estamos aproximando

$$f(x,y)=\operatorname{sen}xy\approx\frac{\pi}{2}x-\frac{\pi^3}{24}x^3$$

(*) $(a+b)^3=a^3+3\cdot a^2b+3\cdot ab^2+b^3$ ou

$$(a+b)^3=(a+b)^2\cdot(a+b)$$

EXERCÍCIO DA LISTA 05:

5. Calcule as diferenciais totais de cada função a seguir:

(a) $z = \dfrac{x}{\sqrt{x^2+y^2}}$

(b) $z = \ln(xy + y^2)$

(c) $z = \arctan \dfrac{x+y}{1-xy}$

(d) $z = \dfrac{ye^x}{\sqrt{x^2+y^2}}$

(e) $z = \arcsin(x\sqrt{1-y^2} + y\sqrt{1-x^2})$

(f) $z = \dfrac{x \sin y}{\cos(xy)}$

$x(1-y^2)^{\frac{1}{2}} + y(1-x^2)^{\frac{1}{2}}$

$y = \operatorname{arcsen} u \Rightarrow y' = ?$

$\Updownarrow$

$u = \operatorname{sen} y$. Derivando em x:

$u' = \cos y \cdot y' \Rightarrow y' = \dfrac{u'}{\cos y}$

$\operatorname{sen}^2 y + \cos^2 y = 1$

$\cos y = \sqrt{1-\operatorname{sen}^2 y} = \sqrt{1-u^2}$

$y' = \dfrac{u'}{\sqrt{1-u^2}}$

Assim:

$$dz = \frac{\partial z}{\partial x} \cdot dx + \frac{\partial z}{\partial y} \cdot dy \quad , \text{ onde:}$$

$$\frac{\partial z}{\partial x} = \frac{\sqrt{1-y^2} + y \cdot \frac{1}{\cancel{2}}(1-x^2)^{-\frac{1}{2}} \cdot (-\cancel{2}x)}{\sqrt{1-\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)^2}}$$

$$= \frac{\sqrt{1-y^2} - \frac{xy}{\sqrt{1-x^2}}}{\sqrt{1-\left[x^2(1-y^2) + 2xy\sqrt{(1-y^2)(1-x^2)} + y^2(1-x^2)\right]}}$$

$$= \frac{\dfrac{xy\sqrt{1-y^2}}{\sqrt{1-x^2}}}{\sqrt{1 - x^2(1-y^2) - 2xy\sqrt{(1-y^2)(1-x^2)} - y^2(1-x^2)}}$$

$$\frac{\partial z}{\partial y} = \dots$$