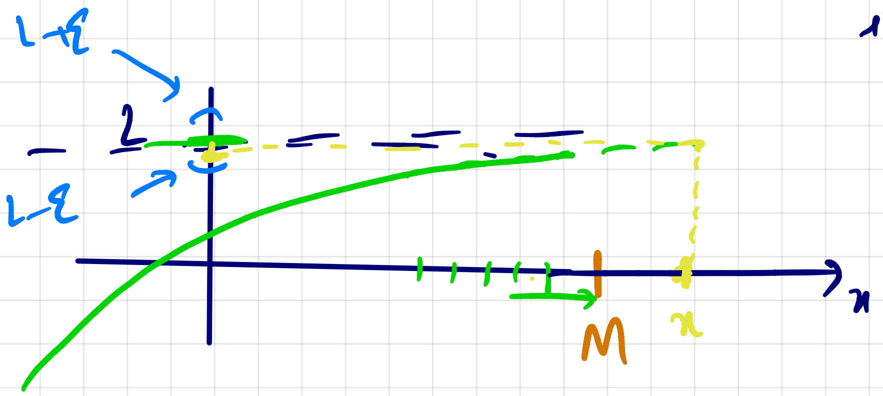
CÁLCULO 1, T2

23/07/24 — AULA 17

## LIMITES NO INFINITO:

**Def.:** $\lim_{x \to +\infty} f(x) = L \overset{def.}{\Longleftrightarrow}$ Dado $\varepsilon > 0$, $\exists M > 0$ tal que, $\forall x > M$, implica em $|f(x) - L| < \varepsilon$.




**Ex.:** $\lim_{x \to +\infty} \frac{1}{x^2} = 0$

Dado $\varepsilon > 0$, precisamos achar $M > 0$, tal que,

$$\forall x > M \Rightarrow |f(x) - 0| < \varepsilon.$$

Analisando $|f(x) - 0|$:

$$|f(x) - 0| = \left|\frac{1}{x^2} - 0\right| = \left|\frac{1}{x^2}\right| = \frac{1}{x^2}$$

Como queremos $x > M$. $\Rightarrow x^2 > M^2$

$\Rightarrow \frac{1}{x^2} < \frac{1}{M^2}$. Assim,

$$|f(x) - 0| = \frac{1}{x^2} < \frac{1}{M^2} = \varepsilon.$$

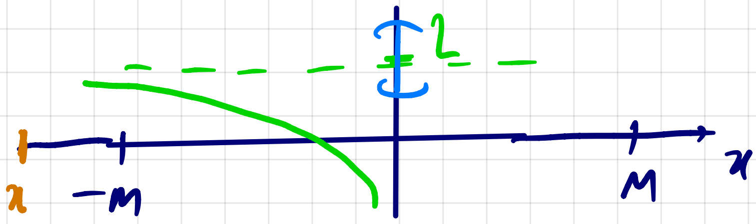
$\longrightarrow M^2 = \frac{1}{\varepsilon} \Rightarrow M = \frac{1}{\sqrt{\varepsilon}}$.

Então, basta tomar $M = \dfrac{1}{\sqrt{\varepsilon}}$.

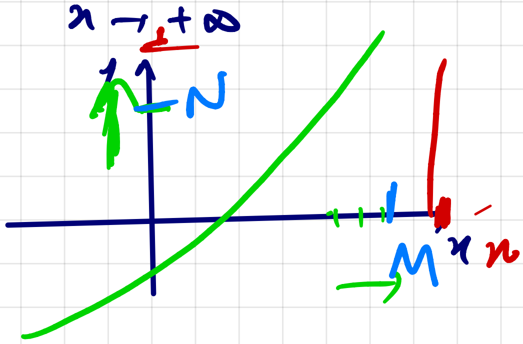
$\square$

De forma similar podemos definir outros limites infinitos:

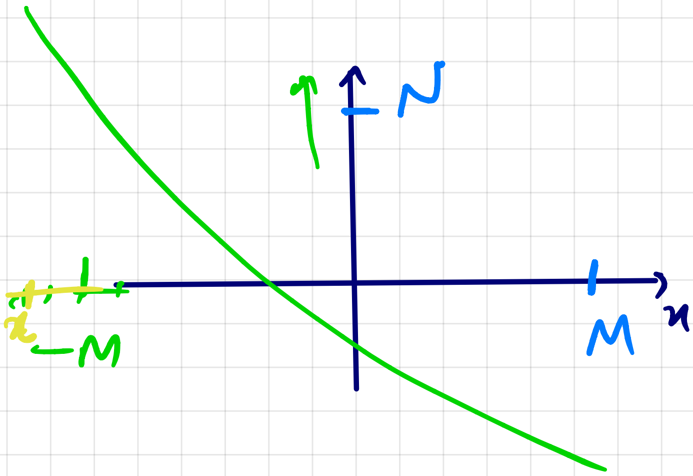
- $\lim\limits_{x \to -\infty} f(x) = L \overset{\text{def.}}{\iff} \forall \varepsilon > 0, \exists M > 0$, tal que. $\forall x < -M \Rightarrow |f(x) - L| < \varepsilon$.



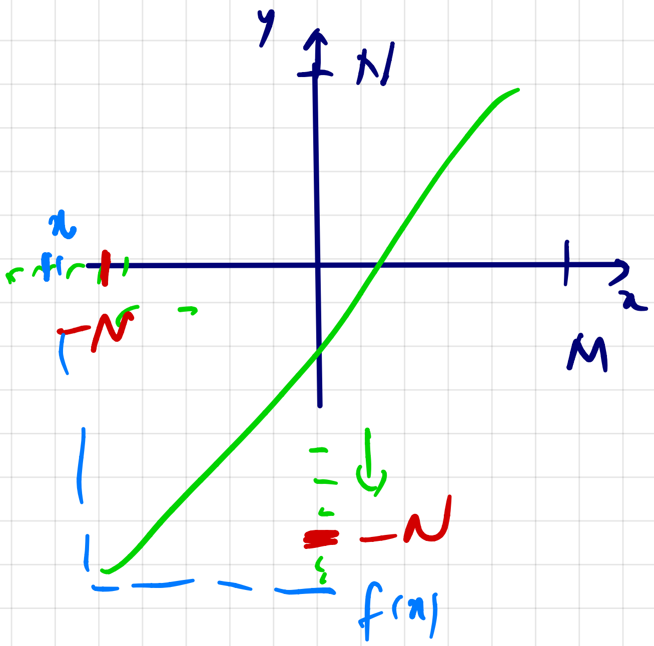

- $\lim\limits_{x \to +\infty} f(x) = +\infty \overset{\text{def.}}{\iff} \forall N > 0, \exists M > 0$, tal que, $\forall x > M \Rightarrow f(x) > N$.




- $\lim\limits_{x \to -\infty} f(x) = +\infty \overset{\text{def.}}{\iff} \forall N > 0, \exists M > 0$ tal que, $\forall x < -M \Rightarrow f(x) > N$



• $\lim\limits_{x \to -\infty} f(x) = -\infty \overset{\text{def.}}{\Longleftrightarrow} \forall N > 0, \exists M > 0,$ tal que $\forall x < -M \Rightarrow f(x) < -N.$




**Ex.:** Mostrar que $\lim\limits_{x \to -\infty} x^3 = -\infty.$

Dado $N > 0$. Precisamos achar $M > 0$, tal que, $\forall x < -M \Rightarrow f(x) < -N.$

Note que:

$$x < -M \Rightarrow \underset{\substack{\parallel \\ f(x)}}{x^3} < (-M)^3 = -M^3 = -N.$$

Então, tome $M = \sqrt[3]{N} > 0.$

Assim:
$$f(x) = x^3 < (-M)^3 = -(M)^3 = -(\sqrt[3]{N})^3 = -N$$
$\square$

INDETERMINAÇÃO DO TIPO $\frac{\infty}{\infty}$.

Analisando $f(x) = \frac{x}{x^2}$ ; $g(x) = \frac{x^2}{x}$ ; $h(x) = \frac{x^2}{x^2}$ ;

então o que resulta

$$\lim_{x \to +\infty} f(x) \; ; \quad \lim_{x \to +\infty} g(x) \text{ e } \lim_{x \to +\infty} h(x) \; ?$$

$$= \frac{\infty}{\infty} \qquad = \frac{\infty}{\infty} \qquad = \frac{\infty}{\infty}$$

Então, $\frac{\infty}{\infty}$ é um SÍMBOLO DE INDETERMINAÇÃO,

pois, note que:

- $\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \frac{\cancel{x}}{x^{\cancel{2}}} = \lim_{x \to +\infty} \frac{1}{x} = 0$

$\frac{1}{x}$ !

$\frac{1}{10} = 0{,}1$

$\frac{1}{100} = 0{,}01$

$\frac{1}{1000} = 0{,}001 \xrightarrow[x \to \infty]{} 0$.

- $\lim_{x \to +\infty} g(x) = \lim_{x \to +\infty} \frac{x^{\cancel{2}}}{\cancel{x}} = \lim_{x \to +\infty} x = +\infty$.

- $\lim\limits_{x\to+\infty} h(x) = \lim\limits_{x\to+\infty} \dfrac{\cancel{x^2}}{\cancel{x^2}} \lim\limits_{x\to+\infty} 1 = 1.$

Na prática, se faz como no exemplo abaixo:

02) $$\lim_{x\to+\infty} \frac{x^3-3x^2-2x+1}{x^4-x^3+x} = \lim_{x\to+\infty} \frac{x^3.\left[1 - \underbrace{\frac{3}{x}}_{\to 0} - \underbrace{\frac{2}{x^2}}_{\to 0} + \underbrace{\frac{1}{x^3}}_{\to 0}\right]}{x^4.\left[1 - \underbrace{\frac{1}{x}}_{\to 0} + \underbrace{\frac{1}{x^3}}_{\to 0}\right]}$$

$$= \lim_{x\to+\infty} \frac{\cancel{x^3}}{\cancel{x^4}} = \lim_{x\to+\infty} \frac{1}{x} = 0$$

"Limpando" a resolução, fazemos:

$$\lim_{x\to+\infty} \frac{x^3-3x^2-2x+1}{x^4-x^3+x} = \lim_{x\to+\infty} \frac{\cancel{x^3}}{\cancel{x^4}} = \lim_{x\to+\infty} \frac{1}{\cancel{x}} = 0 \;//$$

Ilustração gráfica da f.

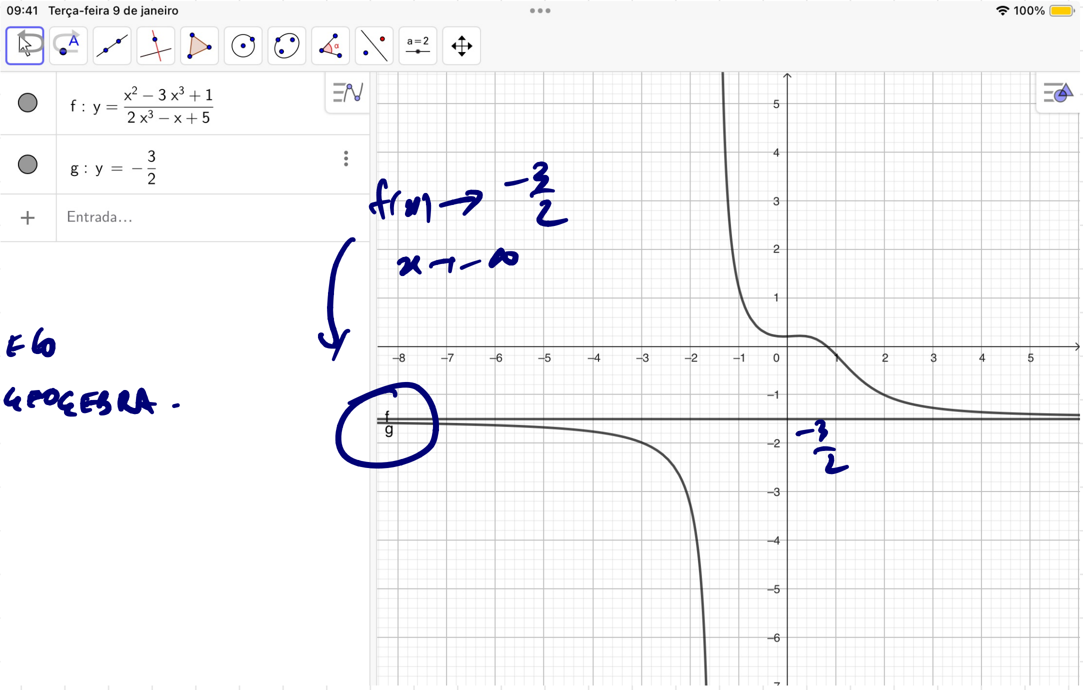

$f(x) \to 0$
$x \to +\infty$

Vejamos outros exemplos:

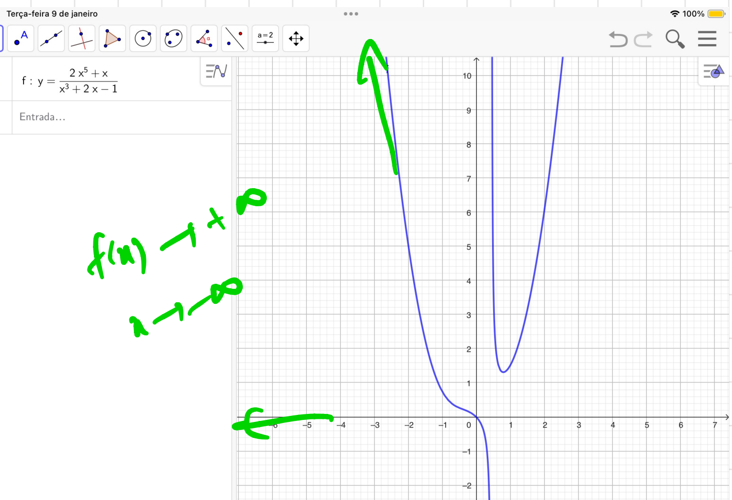
02) $$\lim_{x\to-\infty} \frac{x^2-3x^3+1}{2x^3-x+5} = \lim_{x\to-\infty} \frac{-3\cancel{x^3}}{2\cancel{x^3}} =$$

$$\lim_{x\to-\infty} -\frac{3}{2} = -\frac{3}{2}$$

ilustração pelo geogebra.




03) $$\lim_{x\to-\infty} \frac{2x^5+x}{x^3+2x-1} = \lim_{x\to-\infty} \frac{2x^{\cancel{5}\,2}}{\cancel{x^3}} = \lim_{x\to-\infty} 2x^2$$




$$= 2\cdot(-\infty)^2 = +\infty.$$

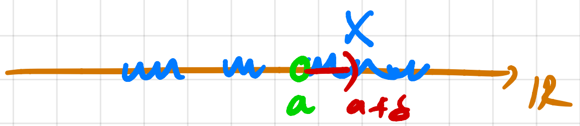
04) $$\lim_{x\to+\infty} \frac{x - x^2 - 3x^3}{2x^2+1} = \lim_{x\to+\infty} \frac{-3x^{\cancel{3}}}{2\cancel{x^2}} =$$

$$= \lim_{x\to+\infty} \frac{-3x}{2} = -\frac{3}{2}\cdot(+\infty) = -\infty$$ //

## LIMITES LATERAIS:

<u>Def</u>: Seja $f: X \setminus \{a\} \to \mathbb{R}$ uma função, onde $a \in \mathbb{R}$ é um PONTO de ACUMULAÇÃO À DIREITA do conjunto $X$

[ ou seja, $\forall \delta > 0$, $(a, a+\delta) \cap X \neq \emptyset$

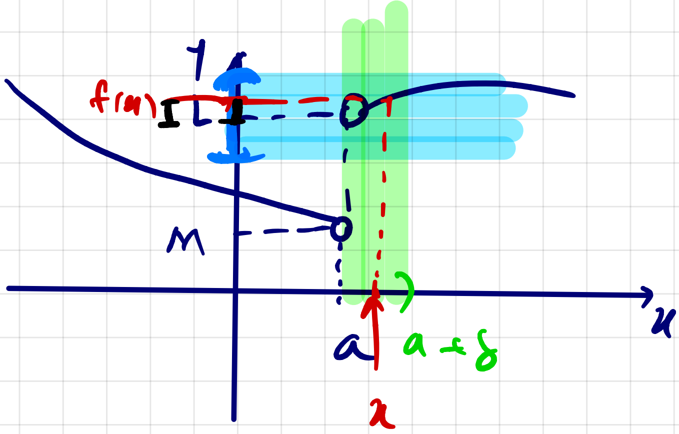


]

Dizemos que $L \in \mathbb{R}$ é o limite da $f(x)$ quando $x$ tende para $a$ pela direita, e escrevemos

$\lim_{x\to a^+} f(x) = L$ se, e somente se

$\forall \varepsilon > 0$, $\exists \delta > 0$, tal que $\forall x \in D(f)$: $a < x < a+\delta$, implique em $|f(x) - L| < \varepsilon$.




$\forall \varepsilon > 0$.
constrói-se o intervalo
$(L-\varepsilon, L+\varepsilon)$

$\exists\, \delta > 0$ tal que;
$\forall x \in (a, a+\delta)$
$\Downarrow$
$|f(x) - L| < \varepsilon$.

Analogamente definimos limite à esquerda:

$$\lim_{x \to a^-} f(x) = L \overset{\text{def.}}{\Longleftrightarrow} \forall\, \varepsilon > 0,\ \exists\, \delta > 0,\ \text{tal que},\ \forall x \in D(f):$$
$$a - \delta < x < a \Longrightarrow |f(x) - L| < \varepsilon.$$

Na ilustração acima:

$$\lim_{x \to a^+} f(x) = L \quad \text{e} \quad \lim_{x \to a^-} f(x) = M.$$

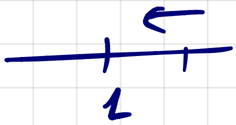
Note que

$$\exists \lim_{x \to a} f(x) \Longleftrightarrow \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

Ou seja, para existir limite em um ponto, os limites laterais devem existir e serem iguais.

Ex.: 01) Como obter $\lim_{x\to 1^+} \frac{1}{x-1}$ ?

Solução: Dado $\delta > 0$. Escreva $x = 1+\delta$.



Então, $x \to 1^+ \Leftrightarrow \delta \to 0^+$

Disso, escrevemos:

$$\lim_{x\to 1^+} \frac{1}{x-1} = \lim_{\delta\to 0^+} \frac{1}{\cancel{1}+\delta-\cancel{1}} = \lim_{\delta\to 0^+} \frac{1}{\underbrace{\delta}_{0^+}} = +\infty$$

$$\frac{1}{0{,}1} = \frac{1}{\frac{1}{10}} = 10$$

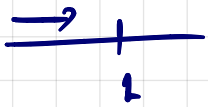
$$\frac{1}{0{,}01} = \frac{1}{\frac{1}{100}} = 100$$

$$\vdots \qquad \downarrow \infty$$

02) $\lim_{x\to 1^-} \frac{x+1}{x-1}$ ?

$$\frac{1+1}{\underline{1-1}} = \frac{2}{\underline{0}} = -\infty.$$

Dado $\delta > 0$.



Escreva $\boxed{x = 1-\delta}$. Então $x \to 1^- \Leftrightarrow \delta \to 0^+$

Disso, obtemos:

$$\lim_{x \to 1^-} \frac{x+1}{x-1} = \lim_{\delta \to 0^+} \frac{1-\delta+1}{\cancel{1}-\delta-\cancel{1}} = \lim_{\delta \to 0^+} \frac{2-\delta}{-\delta}$$

$$= \frac{2}{-0^+} = -\infty$$

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Ex.: Dada $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} x+1, & \text{se } x \geq 1 \\ x^2, & \text{se } x < 1. \end{cases}$

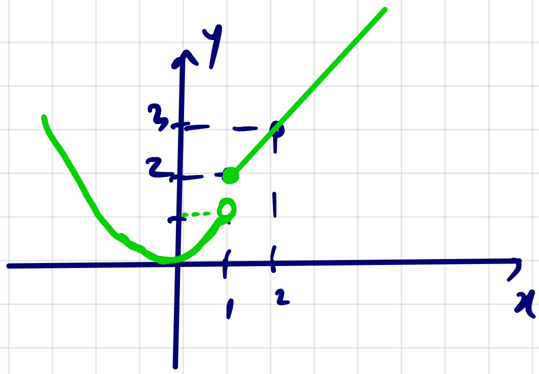
Perguntamos: $\exists \lim_{x \to 1} f(x)$ ?

Solução:

- $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x+1 = 1+1 = 2$

- $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^2 = (1)^2 = 1$

$\Rightarrow \lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$. Conclusão: $\nexists \lim_{x \to 1} f(x)$



Com o estudo de limites laterais e limites no infinito, é possível construir o esboço gráfico de algumas funções.

Ex: $f(x) = \dfrac{2x+1}{x-1}$. Esboço gráfico?

Solução: $D(f) = ?$

$x - 1 \neq 0 \Leftrightarrow x \neq 1$. $\quad D(f) = \mathbb{R} \setminus \{1\}$

$x = 1$ é assíntota vertical.

zeros: $f(x) = 0 \Leftrightarrow \dfrac{2x+1}{x-1} = 0$

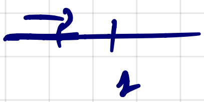
$\Leftrightarrow 2x + 1 = 0 \Leftrightarrow \boxed{x = -\dfrac{1}{2}}$

$\lim\limits_{x\to 1^-} f(x)$ ; $\lim\limits_{x\to 1^+} f(x)$ ; $\lim\limits_{x\to \pm\infty} f(x)$.

- $\lim\limits_{x\to 1^-} f(x)$ :

Dado $\delta > 0$, escreva

$x = 1 - \delta$.



Assim:

$$\lim_{x\to 1^-} \frac{2x+1}{x-1} = \lim_{\delta\to 0^+} \frac{2\cdot(1-\delta)+1}{\cancel{1}-\delta-\cancel{1}} = \lim_{\delta\to 0^+} \frac{2-2\delta+1}{-\delta}$$

$$= \lim_{\delta\to 0^+} \frac{3-2\delta}{-\delta} = \frac{3}{-0^+} = -\infty$$

• $\lim_{x \to 1^+} f(x)$ : Dado $\delta > 0$, escreva : $x = 1 + \delta$. Assim:

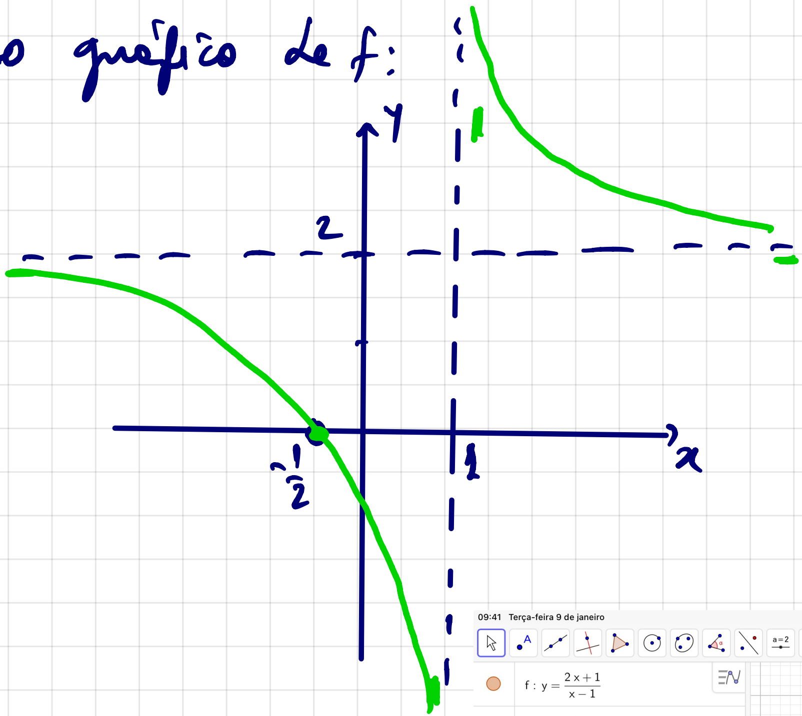
$$\lim_{x \to 1^+} \frac{2x+1}{x-1} = \lim_{\delta \to 0^+} \frac{2.(1+\delta)+1}{\cancel{1}+\delta-\cancel{1}} = \lim_{\delta \to 0^+} \frac{2+2\delta+1}{\delta}$$

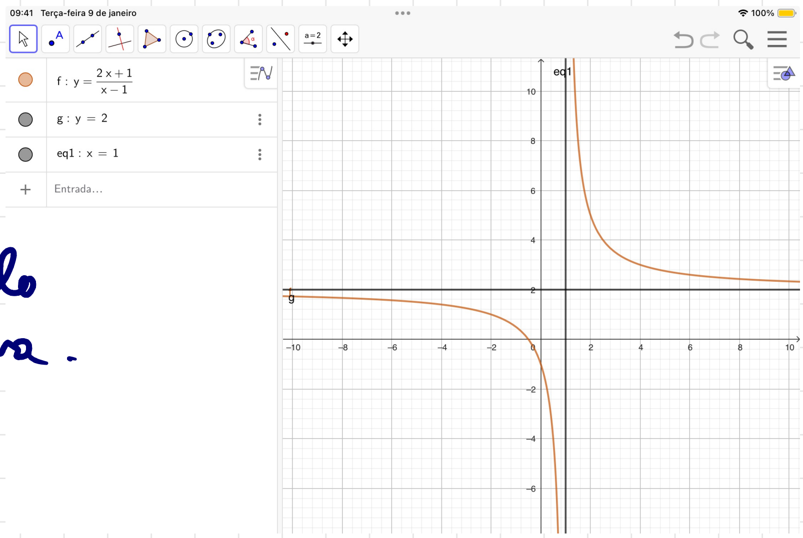
$$= \lim_{\delta \to 0^+} \frac{3+2\delta}{\delta} = \frac{3}{0^+} = +\infty$$

• $\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{2x+1}{x-1} = \lim_{x \to \pm\infty} \frac{2\cancel{x}}{\cancel{x}} = 2$

ASSINTOTA HORIZONTAL

Esboço gráfico de f:




esboço feito pelo geogebra.