CÁLCULO 1, T2

13/07/24 – AULA 14

LISTA 03:

9. Calcule cada limite a seguir, se existir[1]:

(a) $\lim_{x\to 3} \frac{x^3-27}{x-3}$ (b) $\lim_{x\to -1} \frac{x^2-1}{x^2+3x+2}$ (c) $\lim_{x\to 1} \frac{3x^2-4x+1}{x^2+5x-6}$

(d) $\lim_{x\to 1} \frac{x^3-3x+2}{x^4-4x+3}$ (e) $\lim_{x\to a} \frac{x^2-(a+1)x+a}{x^3-a^3}$ (f) $\lim_{x\to a} \frac{x^n-a^n}{x-a}$

(g) $\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4}$ (h) $\lim_{x\to 7} \frac{2-\sqrt{x-3}}{x^2-49}$ (i) $\lim_{x\to 4} \frac{\sqrt{2x+1}-3}{\sqrt{x-2}-\sqrt{2}}$

(j) $\lim_{x\to 4} \frac{3-\sqrt{5+x}}{1-\sqrt{5-x}}$ (k) $\lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}$ (ℓ) $\lim_{x\to -2} \frac{1-\sqrt{x+3}}{\sqrt{x^2+x-1}-1}$

(m) $\lim_{x\to 1} \frac{\sqrt[3]{x+7}-2}{x^2-1}$ (n) $\lim_{x\to 2} \frac{\sqrt{x^2-3}-\sqrt{x-1}}{x^2-4}$ (o) $\lim_{x\to 1} \frac{\sqrt{1-x}}{x^2-3x+2}$

0/0 (INDET.)

(b) $\lim_{x\to -1} \frac{x^2-1}{x^2+3x+2} = \lim_{x\to -1} \frac{\cancel{(x+1)}\cdot(x-1)}{\cancel{(x+1)}\cdot(x+2)} =$

$$
\begin{array}{l|l}
\cancel{x^2}+3x+2 & x+1 \\
\hline
-\cancel{x^2}-x & x+2 \\
\quad 2x+\cancel{2} & \\
\quad -2x\cancel{-2} & \\
\hline
\quad 0 &
\end{array}
$$

$\Rightarrow x^2+3x+2 = (x+1)\cdot(x+2)$

$= \lim_{x\to -1} \frac{x-1}{x+2} = \frac{-1-1}{-1+2} = \frac{-2}{1} = -2$ //

c) $$\lim_{x \to 1} \frac{3x^2 - 4x + 1}{x^2 + 5x - 6} = \frac{0}{0} \quad \text{(INDET.)}$$

$$\begin{array}{rrr|l}
\cancel{3x^2} & -4x & +1 & x - 1 \\ \hline
-\cancel{3x^2} & +3x & & 3x - 1 \\ \hline
 & -x & \cancel{+1} & \\
 & +x & \cancel{-1} & \\ \hline
 & & 0 &
\end{array}$$

$$\Rightarrow 3x^2 - 4x + 1 = (x-1)\cdot(3x-1)$$

$$\begin{array}{rrr|l}
\cancel{x^2} & +5x & -6 & x - 1 \\ \hline
-\cancel{x^2} & +x & & x + 6 \\ \hline
 & 6x & \cancel{-6} & \\
 & -6x & \cancel{+6} & \\ \hline
 & & 0 &
\end{array}$$

$$\Rightarrow x^2 + 5x - 6 = (x-1)\cdot(x+6)$$

Com isso, teremos:

$$\lim_{x \to 1} \frac{3x^2 - 4x + 1}{x^2 + 5x - 6} = \lim_{x \to 1} \frac{\cancel{(x-1)}\cdot(3x-1)}{\cancel{(x-1)}\cdot(x+6)} = \lim_{x \to 1} \frac{3x - 1}{x + 6}$$

$$= \frac{3\cdot(1) - 1}{1 + 6} = \frac{2}{7}$$

h) $\lim_{x \to 7} \dfrac{2-\sqrt{x-3}}{x^2-49} =$ $\dfrac{0}{0}$ (İNDET.)

$(a+b)(a-b) = a^2 - b^2$

$$\lim_{x \to 7} \frac{2-\sqrt{x-3}}{x^2-49} \times \frac{2+\sqrt{x-3}}{2+\sqrt{x-3}} =$$

$$= \lim_{x \to 7} \frac{(2)^2 - (\cancel{\sqrt{x-3}})^{\cancel{2}}}{(x+7)(x-7).(2+\sqrt{x-3})} =$$

$$= \lim_{x \to 7} \frac{4-(x-3)}{(x+7)(x-7)(2+\sqrt{x-3})} =$$

$$= \lim_{x \to 7} \frac{7-x}{(x+7)(x-7)(2+\sqrt{x-3})} =$$

$$= \lim_{x \to 7} \frac{-\cancel{(x-7)}}{(x+7)\cancel{(x-7)}.(2+\sqrt{x-3})} =$$

$$= \lim_{x \to 7} \frac{-1}{(x+7)(2+\sqrt{x-3})} = \frac{-1}{(7+7)(2+\sqrt{7-3})}$$

$$= \frac{-1}{14.(2+2)} = -\frac{1}{56}$$

j) $\lim_{x \to 4} \dfrac{3-\sqrt{5+x}}{1-\sqrt{5-x}} =$ $\frac{0}{0}$ (INDET.)

$$= \lim_{x \to 4} \frac{3-\sqrt{5+x}}{1-\sqrt{5-x}} \times \frac{3+\sqrt{5+x}}{3+\sqrt{5+x}} \cdot \frac{1+\sqrt{5-x}}{1+\sqrt{5-x}} =$$

$$= \lim_{x \to 4} \frac{\left[(3)^2 - (\sqrt{5+x})^2\right] \cdot (1+\sqrt{5-x})}{\left[1^2 - (\sqrt{5-x})^2\right] \cdot (3+\sqrt{5+x})} =$$

$$= \lim_{x \to 4} \frac{(9-5-x)(1+\sqrt{5-x})}{(1-5+x) \cdot (3+\sqrt{5+x})} = \lim_{x \to 4} \frac{(4-x)(1+\sqrt{5-x})}{(-4+x)(3+\sqrt{5+x})}$$

$$= \lim_{x \to 4} \frac{(\cancel{4-x})(1+\sqrt{5-x})}{-(\cancel{4-x})(3+\sqrt{5+x})} = \lim_{x \to 4} \frac{1+\sqrt{5-x}}{-(3+\sqrt{5+x})} =$$

$$= \frac{1+\sqrt{5-4}}{-(3+\sqrt{5+4})} = \frac{2}{-(3+3)} = -\frac{2}{6} = -\frac{1}{3}$$

L3:

5. Mostre que se $\lim_{x\to 3} xf(x) = 12$, então existe $\lim_{x\to 3} f(x)$ e é igual a 4.

$$12 = \lim_{x\to 3} x \cdot f(x)$$

$$\lim_{x\to a} f(x)\cdot g(x) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$$

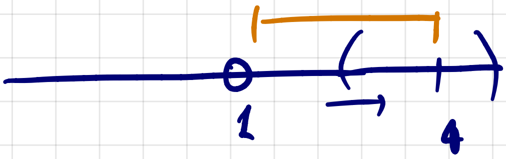
$$12 = \lim_{x\to 3} x\cdot f(x) = \underbrace{\lim_{x\to 3} x}_{=3} \cdot \lim_{x\to 3} f(x)$$

$$12 = 3 \cdot \lim_{x\to 3} f(x) \Rightarrow \lim_{x\to 3} f(x) = \frac{12}{3} = 4$$

L3: 4. Usando a definição de limite, prove que

(a) $\lim_{x\to 2} x^2 = 4$ (b) $\lim_{x\to 4} \frac{1}{x-1} = \frac{1}{3}$ (c) $\lim_{x\to a} \sqrt{x} = \sqrt{a}$, $a > 0$ (d) $\lim_{x\to 0} x \operatorname{sen} \frac{1}{x} = 0$.

(b) $\lim_{x\to 4} \frac{1}{x-1} = \frac{1}{3}$



Dado $\varepsilon > 0$, precisamos achar $\delta > 0$, tal que,

$0 < \delta < 3$

PARA EVITAR QUE 1 FIQUE NO INTERVALO

$\forall x \in D(f)$: $0 < |x-4| < \delta$, implique em

$$\left|f(x) - \frac{1}{3}\right| < \varepsilon.$$

Avaliando $\left|f(x) - \frac{1}{3}\right|$, temos:

$$\left|f(x)-\frac{1}{3}\right| = \left|\frac{1}{x-2} - \frac{1}{3}\right| = \left|\frac{3-(x-1)}{3.(x-1)}\right| = \left|\frac{3-x+1}{3(x-1)}\right|$$

$$= \left|\frac{4-x}{3(x-1)}\right| = \frac{\overbrace{|x-4|}^{<\delta}}{3.|x-1|} < \frac{\delta}{3.|x-1|} \qquad (*)$$

Note que:

$$\underbrace{|x-1|}_{} = |1-x| = |1-x+4-4| = |-3-x+4|$$

$$= |-3-(x-4)| \geq |-3| - |x-4|$$

$$\boxed{|a \pm b| \geq |a| - |b|}$$

$$= 3 - |x-4| > 3-\delta > 0$$

$$|x-4| < \delta$$

$$-|x-4| > -\delta$$

pois $0 < \delta < 3$

$$\Rightarrow |x-1| > 3-\delta \Rightarrow \frac{1}{|x-1|} < \frac{1}{3-\delta}$$

Assim, teremos de $(*)$ que:

$$\left|f(x)-\frac{1}{3}\right| < \frac{\delta}{3.|x-1|} = \frac{\delta}{3} \cdot \underbrace{\frac{1}{|x-1|}}_{<\frac{1}{3-\delta}} < \frac{\delta}{3} \cdot \frac{1}{3-\delta} = \varepsilon$$

Ou seja, obtemos a relação:

$$\left.\begin{array}{l} \dfrac{\delta}{3(3-\delta)} = \varepsilon \\[2ex] \dfrac{\delta}{3-\delta} = 3\varepsilon \end{array}\right\}$$

$$\Longrightarrow \quad \delta = 3\varepsilon(3-\delta)$$

$$\delta = 9\varepsilon - 3\delta\varepsilon$$

$$\delta + 3\delta\varepsilon = 9\varepsilon$$

$$\delta(1+3\varepsilon) = 9\varepsilon \quad \Rightarrow \quad \boxed{\delta = \frac{9\varepsilon}{1+3\varepsilon}}$$

Portanto, vale o limite dado

(d) $\lim_{x\to 0} x \cdot \operatorname{sen}\frac{1}{x} = 0$:

Dado $\varepsilon>0$, precisamos encontrar $\delta>0$, tal que,

$\forall x \in D(f)$: $0<|x-0|<\delta \Longrightarrow |f(x)-0|<\varepsilon$

Analisando $|f(x)-0|$:

$$|f(x)-0| = |f(x)| = \left| x \cdot \operatorname{sen}\frac{1}{x} \right| = |x| \cdot \underbrace{\left|\operatorname{sen}\frac{1}{x}\right|}_{|\operatorname{sen} x| \leq 1,\ \forall x.} \leq |x| \cdot 1$$

$$= |x| = |x-0| < \delta = \varepsilon .$$

Ou seja, basta tomar $\delta = \varepsilon$.

$\square$

L3

6. Dê um exemplo em que $\lim_{x\to 0}(f(x)+g(x))$ existe mas nem $\lim_{x\to 0} f(x)$ e nem $\lim_{x\to 0} g(x)$ existem.

solução: Tem-se a seguinte propriedade aritmética:

$$\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) \,;$$

existindo $\lim_{x\to a} f(x)$ e $\lim_{x\to a} g(x)$.

Para este exercício, considere

$$f(x) = \frac{1}{x} \quad \text{e} \quad g(x) = -\frac{1}{x}.$$

Então, $f(x)+g(x) = \frac{1}{x} - \frac{1}{x} = 0$, e disso,

$$\lim_{x\to 0}(f(x)+g(x)) = \lim_{x\to 0} 0 = 0 \,;$$

mas: $\lim_{x\to 0} \frac{1}{x} = \nexists$ e $\lim_{x\to 0} -\frac{1}{x} = \nexists$.