CÁLCULO 1, T2.

11/07/24 – AULA 10

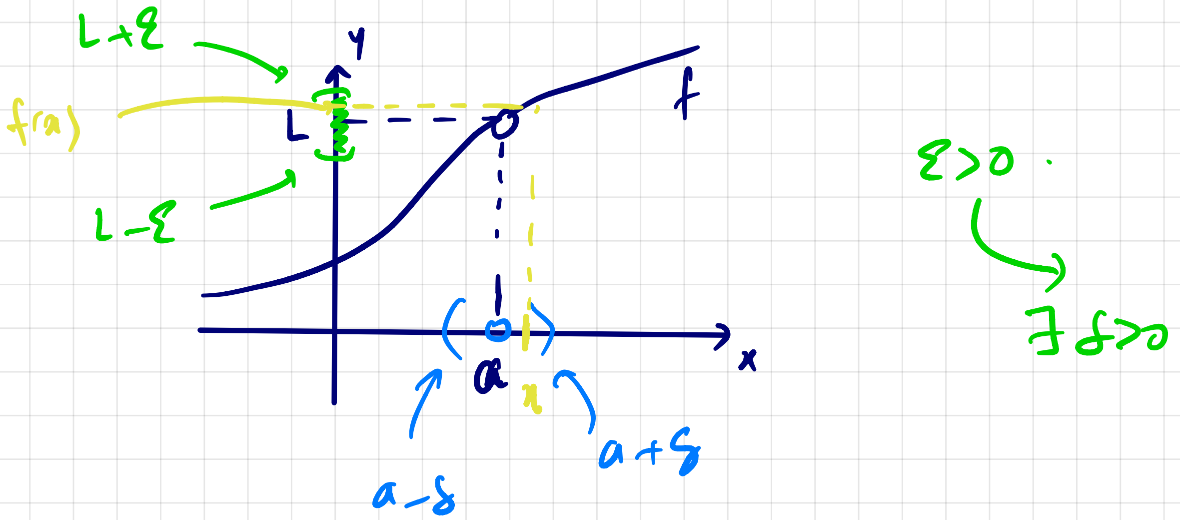
Obs.: AULA EXTRA, PELO E-AULA: sábado, a partir das 16:30

(aula de exercícios, será gravada, podendo ser vista a posteriori)

Na aula passada iniciamos o estudo de limites de funções:

$\lim_{x \to a} f(x) = L \overset{\text{def.}}{\iff} \forall \varepsilon > 0, \exists \delta > 0$, tal que: $\forall x \in D(f)$:

$$0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon.$$



Seguiremos com os exemplos:

02) Mostre que $\lim_{x \to -2} \dfrac{x^2 - 4}{x + 2} = -4$.

$(f: \mathbb{R} \setminus \{-2\} \to \mathbb{R}.)$

SOLUÇÃO: Dado $\varepsilon > 0$, precisamos achar $\delta > 0$ tal que, $\forall x \in D(f)$: $0 < |x - (-2)| < \delta \implies |f(x) - (-4)| < \varepsilon$.

Ou seja, devemos ter $|f(x) + 4| < \varepsilon$, sempre que

$0 < \underbrace{|x+2| < \delta}.$

Analisando $|f(x)+4|$:

$$|f(x)+4| = \left| \frac{x^2-4}{x+2} + 4 \right| = \left| \frac{(\cancel{x+2})(x-2)}{\cancel{x+2}} + 4 \right| =$$

$$= |x-2+4| = \underbrace{|x+2|}_{<\delta} < \delta = \varepsilon.$$

Ou seja, basta tomar $\delta = \varepsilon$.

$\square$

Ilustrando: $\varepsilon = 0{,}1$. Então, $\delta = \varepsilon = 0{,}1$.

Assim, $\forall x \in \mathbb{R} \setminus \{-2\}$: $0 < |x+2| < 0{,}1$,

é garantido que $|f(x)+4| < 0{,}1 = \varepsilon$.

Ex.: tome $x = -2{,}01$. Então:

$$|x+2| = |-2{,}01+2| = |-0{,}01| = 0{,}01 < 0{,}1 = \delta.$$

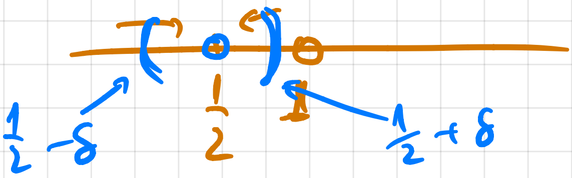
Então, é garantido que $|f(x)+4| < 0{,}1 = \varepsilon$:

$$|f(x)+4| = \left| \frac{(-2{,}01)^2-4}{-2{,}01+2} + 4 \right| =$$

$$= \left| \frac{+0{,}0401}{-0{,}01} + 4 \right| = |-4{,}01+4| = 0{,}01 < 0{,}1 = \varepsilon$$

03) Seja $f: \mathbb{R}\setminus\{\frac{1}{2}, 1\} \to \mathbb{R}$, $f(x) = \dfrac{1}{x-1}$,

prove que $\displaystyle\lim_{x \to \frac{1}{2}} \frac{1}{x-1} = -2$.



<u>solução:</u> Dado $\varepsilon > 0$, precisamos achar $\delta > 0$, $0 < \delta < \frac{1}{2}$,

(para evitar que o intervalo $(\frac{1}{2}-\delta, \frac{1}{2}+\delta)\setminus\{\frac{1}{2}\}$ fique com o ponto 1 em seu interior, pois f não está definida ali)

tal que, $\forall x \in D(f): 0 < |x - \frac{1}{2}| < \delta \Longrightarrow |f(x) - (-2)| < \varepsilon$.

Analisando $|f(x) + 2|$:

$$|f(x)+2| = \left|\frac{1}{x-1} + 2\right| = \left|\frac{1 + 2.(x-1)}{x-1}\right| =$$

$$= \left|\frac{1+2x-2}{x-1}\right| = \left|\frac{2x-1}{x-1}\right| = \frac{|2x-1|}{|x-1|} =$$

$$= \frac{|2(x-\frac{1}{2})|}{|x-1|} = \frac{2.\overbrace{|x-\frac{1}{2}|}^{<\delta}}{|x-1|} < \frac{2\delta}{|x-1|}$$

Note que $|x-1| = |1-x| = \left|\frac{1}{2} + \left(\frac{1}{2} - x\right)\right| =$

$$= \left|\frac{1}{2} + \left(\frac{1}{2}-x\right)\right| \geq \frac{1}{2} \overbrace{- \left|x - \frac{1}{2}\right|}^{> -\delta} > \frac{1}{2} - \delta \overset{\text{pois } \delta < \frac{1}{2}}{>} 0$$

$|a \pm b| \geq |a| - |b|$ → AULA <u>03</u>

$$|x-\tfrac{1}{2}|<\delta \Rightarrow -|x-\tfrac{1}{2}|>-\delta$$

i.e.,

$$|x-1|>\frac{1}{2}-\delta>0 \Longrightarrow \frac{1}{|x-1|}<\frac{1}{\frac{1}{2}-\delta}$$

Assim, teremos:

$$|f(x)+2|<\frac{2\delta}{|x-1|}=2\delta\cdot\frac{1}{|x-1|}<2\delta\cdot\frac{1}{\frac{1}{2}-\delta}$$

$$=\frac{2\delta}{\frac{1}{2}-\delta}:=\varepsilon$$

$$\Leftrightarrow\ 2\delta=\varepsilon\cdot\left(\frac{1}{2}-\delta\right)$$

$$2\delta=\frac{\varepsilon}{2}-\varepsilon\cdot\delta$$

$$2\delta+\varepsilon\cdot\delta=\frac{\varepsilon}{2}$$

$$\delta(2+\varepsilon)=\frac{\varepsilon}{2}$$

$$\Rightarrow\ \delta=\frac{\varepsilon}{2(2+\varepsilon)}$$

Isto mostra que $\lim\limits_{x\to\frac{1}{2}}\frac{1}{x-1}=-2$.

ilustrando: $\varepsilon=0{,}1$. Então: $\delta=\frac{0{,}1}{2\cdot(2+0{,}1)}=\frac{0{,}1}{4{,}2}$

$\delta\cong 0{,}02\cancel{3}80952$

$\delta\cong 0{,}024$.

Ou seja; $\forall x \in D(f)$: tal que

$$0 < \left|x - \frac{1}{2}\right| < 0{,}024 \,, \text{ é}$$

garantido que

$$|f(x) + 2| < 0{,}1$$

DE FATO, POR EX.: $x = 0{,}4993$, é TAL QUE:

$$0 < \left|0{,}4993 - \frac{1}{2}\right| = 0{,}0007 < 0{,}024.$$

Então:

$$|f(x) + 2| = \left|\frac{1}{0{,}4993 - 1} + 2\right| = \left|-1{,}9972039\ldots + 2\right|$$

$$\cong 0{,}002796 < 0{,}1 = \varepsilon.$$

04) Prove que $\lim\limits_{x \to 2} \sqrt{x+1} = \sqrt{3}$; onde

$$f : (-1, +\infty) \setminus \{2\} \to \mathbb{R}.$$

SOLUÇÃO: Dado $\varepsilon > 0$, precisamos achar $\delta > 0$, tal que, $\forall x \in D(f)$ tal que $0 < |x - 2| < \delta \implies |f(x) - \sqrt{3}| < \varepsilon$.

Avaliando $|f(x) - \sqrt{3}|$:

$$(a-b)(a+b) = a^2 - b^2$$

$$|f(x) - \sqrt{3}| = |\sqrt{x+1} - \sqrt{3}| = \left|(\sqrt{x+1} - \sqrt{3}) \cdot \frac{(\sqrt{x+1} + \sqrt{3})}{(\sqrt{x+1} + \sqrt{3})}\right| =$$

RACIONALIZANDO:

$$= \left| \frac{(\sqrt{x+1})^2 - (\sqrt{3})^2}{\sqrt{x+1} + \sqrt{3}} \right| = \frac{|x+1-3|}{|\sqrt{x+1} + \sqrt{3}|} =$$

$$= \frac{\overbrace{|x-2|}^{<\delta}}{|\sqrt{x+1} + \sqrt{3}|} < \frac{\delta}{|\sqrt{x+1} + \sqrt{3}|}$$

Note que:

$$|\sqrt{x+1} + \sqrt{3}| \geq \underbrace{\sqrt{x+1}}_{\geq 0} + \sqrt{3} \geq \sqrt{3}$$

$$\Longrightarrow \frac{1}{|\sqrt{x+1} + \sqrt{3}|} \leq \frac{1}{\sqrt{3}}$$

Portanto, teremos a estimativa:

$$|f(x) - \sqrt{3}| < \frac{\delta}{|\sqrt{x+1} + \sqrt{3}|} = \delta \cdot \underbrace{\frac{1}{|\sqrt{x+1} + \sqrt{3}|}}_{\leq \frac{1}{\sqrt{3}}} \leq \frac{\delta}{\sqrt{3}} := \varepsilon$$

Então, basta tomar $\boxed{\delta = \sqrt{3} \cdot \varepsilon}$

PROPRIEDADES DOS LIMITES:

PROPOSIÇÃO: Se $\lim_{x \to a} f(x) = L$ e $\lim_{x \to a} g(x) = M$,

então:

(i) $\lim_{x \to a} (f(x) + g(x)) = L + M$ $\left(\text{i.e., } \lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) \right)$

(ii) $\lim_{x \to a} (f(x) - g(x)) = L - M.$

(iii) $\lim_{x \to a} f(x) \cdot g(x) = L \cdot M.$

(iv) $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$, desde que $M \neq 0$.

(v) $\lim_{x \to a} [f(x)]^{g(x)} = L^M.$

Ex.: $\lim_{x \to 2} 3x^2 - x = \lim_{x \to 2} 3x^2 - \lim_{x \to 2} x =$

$$= 3 \cdot (2)^2 - 2 = 3 \cdot 4 - 2 = 10$$

INDETERMINAÇÃO DO TIPO $\frac{0}{0}$:

No cálculo de $\lim\limits_{x\to a} \frac{f(x)}{g(x)}$, se, substituindo x por a, obtemos o símbolo $\frac{0}{0}$, que é indeterminado, significa que aparece o fator $x-a$ no numerador e no denominador. Então, o mesmo deve ser simplificado. Vejamos exemplos:

01) $\lim\limits_{x\to 1} \dfrac{x^2-x}{x^3+2x-3} = \dfrac{0}{0}$ (INDETERM.)

↳ APARECE O FATOR $x-1$ NO NUM. E NO DENOM.

Vamos dividir numerador e denom. por $x-1$.

$$
\begin{array}{r|l}
x^2 - \cancel{x} & x-1 \\ \cline{2-2}
-\cancel{x^2} + x & x \\ \cline{1-1}
0 &
\end{array}
$$

Logo: $x^2-x = x\cdot(x-1)$

$$
\begin{array}{r|l}
\cancel{x^3} + 2x - 3 & x-1 \\ \cline{2-2}
-\cancel{x^3} + x^2 & x^2+x+3 \\ \cline{1-1}
\cancel{x^2} + 2x - 3 & \\
-\cancel{x^2} + x & \\ \cline{1-1}
3x - \cancel{3} & \\
-3x + 3 & \\ \cline{1-1}
0 &
\end{array}
$$

$\Rightarrow x^3+2x-3 = (x-1)\cdot(x^2+x+3)$

Assim, teremos:

$$\lim_{x \to 1} \frac{x^2 - x}{x^3 + 2x - 3} = \lim_{x \to 1} \frac{x \cdot \cancel{(x-1)}}{\cancel{(x-1)} \cdot (x^2 + x + 3)} =$$

$$= \lim_{x \to 1} \frac{x}{x^2 + x + 3} = \frac{1}{1 + 1 + 3} = \frac{1}{5}$$

02) $\displaystyle\lim_{x \to -3} \frac{x^2 - 9}{x^2 + 2x - 3} = \frac{0}{0}$ (INDET.)

$$\begin{array}{rr|l}
x^2 & -9 & x+3 \\ \hline
-x^2 - 3x & & x-3 \\ \hline
-3x & -9 & \\
+3x & +9 & \\ \hline
 & 0 &
\end{array}
\implies x^2 - 9 = (x+3) \cdot (x-3)$$

$$\begin{array}{rr|l}
x^2 + 2x & -3 & x+3 \\ \hline
-x^2 - 3x & & x-1 \\ \hline
-x & -3 & \\
+x & +3 & \\ \hline
 & 0 &
\end{array}
\implies x^2 + 2x - 3 = (x+3)(x-1)$$

Com isso, vamos obter:

$$\lim_{x \to -3} \frac{x^2 - 9}{x^2 + 2x - 3} = \lim_{x \to -3} \frac{\cancel{(x+3)} \cdot (x-3)}{\cancel{(x+3)} \cdot (x-1)} = \lim_{x \to -3} \frac{x-3}{x-1} =$$

$$= \frac{-3-3}{-3-1} = \frac{-6}{-4} = \frac{6}{4} = \frac{3}{2}$$

03) $$\lim_{x \to 2} \frac{\sqrt{x-1} - 1}{x^2 - 4} = \frac{0}{0}$$ (INDET.)

PORÉM AGORA ENVOLVE UM RADICAL. PRECISAMOS RACIONALIZAR.

EX.: $(\sqrt{a} - b) \cdot (\sqrt{a} + b) =$
$(\sqrt{a})^2 - b^2$
$a - b^2$

$$\lim_{x \to 2} \frac{\sqrt{x-1} - 1}{x^2 - 4} \times \frac{\sqrt{x-1} + 1}{\sqrt{x-1} + 1} = \lim_{x \to 2} \frac{(\cancel{\sqrt{x-1}})^{\cancel{2}} - (1)^2}{(x^2-4)(\sqrt{x-1} + 1)}$$

$$= \lim_{x \to 2} \frac{x - 1 - 1}{(x+2)(x-2) \cdot (\sqrt{x-1} + 1)} =$$

$$= \lim_{x \to 2} \frac{\cancel{x-2}}{(x+2)\cancel{(x-2)} \cdot (\sqrt{x-1} + 1)} = \lim_{x \to 2} \frac{1}{(x+2) \cdot (\sqrt{x-1} + 1)}$$

$$= \frac{1}{(2+2) \cdot (\sqrt{2-1} + 1)} = \frac{1}{4 \cdot (2)} = \frac{1}{8}$$

04) $$\lim_{x\to 1} \frac{\sqrt{x^2+3x+1}-\sqrt{6x-1}}{x^3-1} = \frac{0}{0} \quad (\text{INDET.})$$

$$\lim_{x\to 1} \frac{\sqrt{x^2+3x+1}-\sqrt{6x-1}}{x^3-1} \times \frac{\sqrt{x^2+3x+1}+\sqrt{6x-1}}{\sqrt{x^2+3x+1}+\sqrt{6x-1}} =$$

$$= \lim_{x\to 1} \frac{\left(\cancel{\sqrt{x^2+3x+1}}\right)^{\cancel{2}} - \left(\cancel{\sqrt{6x-1}}\right)^{\cancel{2}}}{(x^3-1)\cdot\left(\sqrt{x^2+3x+1}+\sqrt{6x-1}\right)} =$$

$$= \lim_{x\to 1} \frac{x^2+3x+1-6x+1}{(x^3-1)\left(\sqrt{x^2+3x+1}+\sqrt{6x-1}\right)} =$$

$$= \lim_{x\to 1} \frac{x^2-3x+2}{(x^3-1)\left(\sqrt{x^2+3x+1}+\sqrt{6x-1}\right)}$$

$$\begin{array}{rr|l}
\cancel{x^2} & -3x+2 & x-1 \\ \cline{3-3}
-\cancel{x^2} & +x & x-2 \\ \cline{1-2}
 & -2x+\cancel{2} & \\
 & +2x-\cancel{2} & \\ \cline{2-2}
 & 0 &
\end{array}$$

$$\Longrightarrow x^2-3x+2 = (x-1)\cdot(x-2)$$

$$
\begin{array}{l|l}
\cancel{x^3} - 1 & x - 1 \\
\hline
-\cancel{x^3} + x^2 & x^2 + x + 1 \\
\cancel{x^2} - 1 & \\
-\cancel{x^2} + x & \\
x - 1 & \\
-x + 1 & \\
0 &
\end{array}
$$

$$\Longrightarrow x^3 - 1 = (x^2 + x + 1)(x - 1)$$

Portanto:

$$\Leftrightarrow \lim_{x \to 1} \frac{\cancel{(x-1)} \cdot (x-2)}{(x^2+x+1)\cancel{(x-1)}\left(\sqrt{x^2+3x+1} + \sqrt{6x-1}\right)}$$

$$= \frac{1 - 2}{(1^2 + 1 + 1) \cdot \left(\sqrt{1+3+1} + \sqrt{6 \cdot 1 - 1}\right)}$$

$$= \frac{-1}{3 \cdot (2\sqrt{5})} = -\frac{1}{6\sqrt{5}}$$