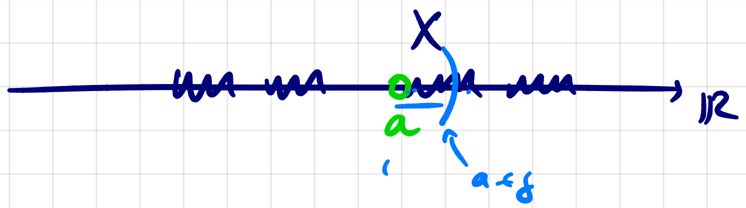
CÁLCULO 1

23/07/24 – AULA 20

## LIMITES LATERAIS:

**Def.:** Seja $f: X \setminus \{a\} \to \mathbb{R}$ uma função e $a \in \mathbb{R}$ um ponto de acumulação à direita do conjunto $X$, ou seja,

$\forall \delta > 0, \quad X \cap (a, a+\delta) \neq \emptyset$

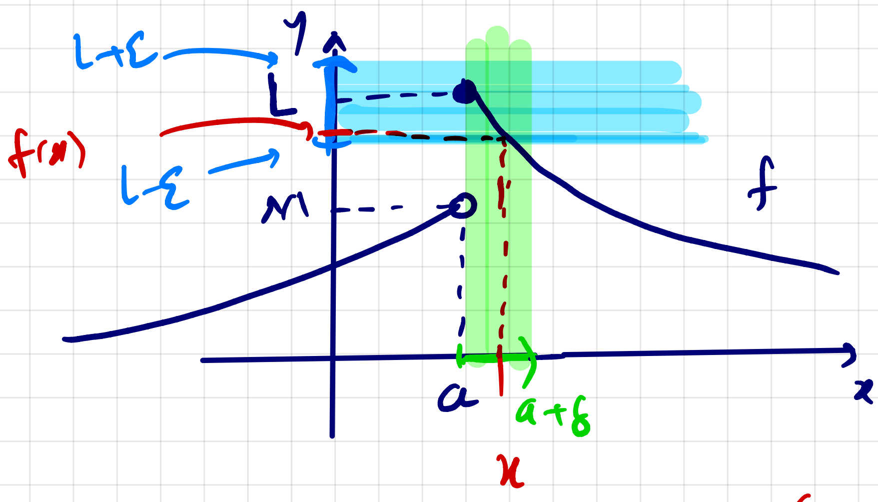



Dizemos que $L \in \mathbb{R}$ é o limite de $f(x)$ quando $x$ tende para $a$ pela direita, e escrevemos

$$\lim_{x \to a^+} f(x) = L,$$

se, e somente se, $\forall \varepsilon > 0, \exists \delta > 0$, tal que,

$\forall x \in D(f): \quad a < x < a + \delta \implies |f(x) - L| < \varepsilon.$




$\forall \varepsilon > 0 \longrightarrow \exists \delta > 0 \quad \forall x \in (a, a+\delta)$

De forma análoga definimos o LIMITE À ESQUERDA de um ponto a.

Dado a um ponto de acumulação à esquerda de um conj. X (i.e., $\forall \delta > 0$, $(a-\delta, a) \cap X \neq \emptyset$)

Então, $\lim_{x \to a^-} f(x) = L$ se, e somente se,

$\forall \varepsilon > 0$, $\exists \delta > 0$, tal que, $\forall x \in D(f)$: $a - \delta < x < a \Rightarrow$

$\Rightarrow |f(x) - L| < \varepsilon$.

No desenho acima, teremos que

$$\lim_{x \to a^-} f(x) = M.$$

Pelo que temos segue que

$$\exists \lim_{x \to a} f(x) = L \iff \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$$

Ou seja, para existir o limite em um ponto, os limites laterais devem existir e devem ser iguais.

Ex.: $f: \mathbb{R} \to \mathbb{R}$, a função dada por

$$f(x) = \begin{cases} x + 1 & \text{se } x \geq 1 \\ x^2 + x & \text{se } x < 1 \end{cases}$$

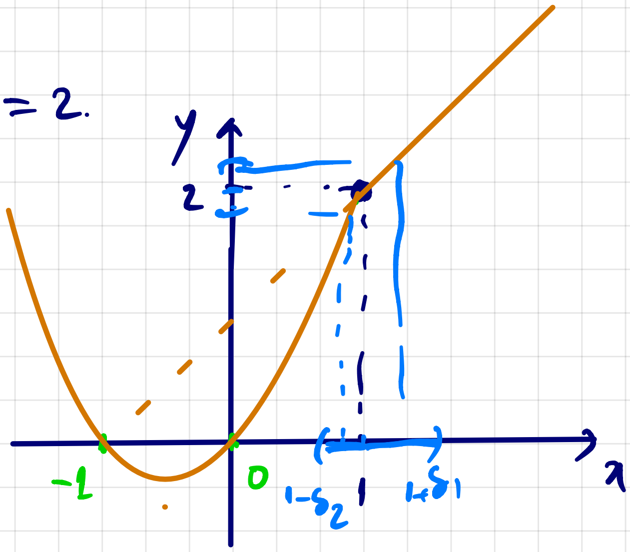
Perguntamos: $\exists \lim_{x\to 1} f(x)$ ?

Solução: • $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x+1 = 1+1 = 2$

• $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2+x = (1)^2+1 = 2$

Como $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = 2$, segue que

$\exists \lim_{x\to 1} f(x) = 2.$




Limites infinitos:

O que resulta, • $\lim_{x\to 1^+} \dfrac{1}{x-1}$ ?

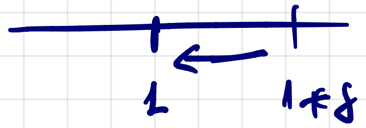
• $\lim_{x\to 1^-} \dfrac{1}{x-1}$ ?

• $\lim_{x\to 1^+} \frac{1}{x-1}$ : tome $\delta>0$ e escreva

$x = 1+\delta$

Note que

$\delta \to 0 \iff x \to 1^+$

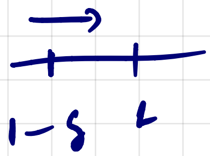



Assim, temos:

$$\lim_{x\to 1^+} \frac{1}{x-1} = \lim_{\delta\to 0^+} \frac{1}{\cancel{1}+\delta-\cancel{1}} = \lim_{\delta\to 0^+} \frac{1}{\delta} = +\infty$$

• $\lim_{x\to 1^-} \frac{1}{x-1}$. Tome $\delta>0$. Escreva

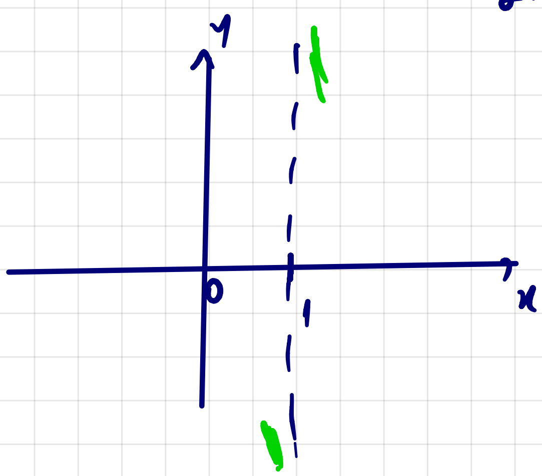
$x = 1-\delta$.




Assim:

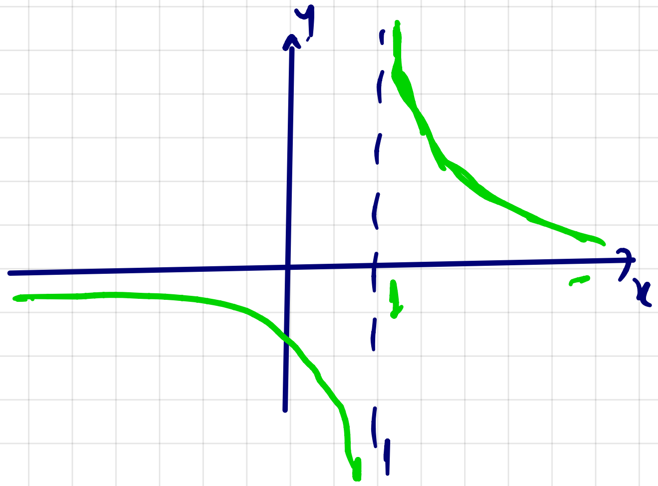
$$\lim_{x\to 1^-} \frac{1}{x-1} = \lim_{\delta\to 0^+} \frac{1}{\cancel{1}-\delta-\cancel{1}} = \lim_{\delta\to 0^+} \frac{1}{-\delta}$$

$$= \lim_{\delta\to 0^+} -\left(\frac{1}{\delta}\right) = -(+\infty) = -\infty$$



Além disso:

$$\lim_{x\to +\infty} \frac{1}{x-1} = \lim_{x\to+\infty} \frac{1}{\underbrace{x}_{+\infty}} = 0$$




Com o auxílio de limites laterais e limites no infinito, podemos traçar o esboço gráfico de algumas funções não vistas no ensino médio. Vejamos exemplos:

Ex: Com a ajuda de limites laterais e no infinito, esboce o gráfico das funções:

(a) $f(x) = \dfrac{x-1}{1+2x}$.

Solução: $D(f) = ?$

$$1+2x \neq 0 \iff x \neq -\frac{1}{2}$$

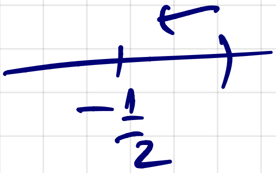
$$D(f) = \mathbb{R} \setminus \{-\tfrac{1}{2}\}.$$

Zeros: $f(x) = 0 \iff \dfrac{x-1}{1+2x} = 0 \iff x-1 = 0 \iff \boxed{x = 1}$

- $\lim\limits_{x\to -\frac{1}{2}^+} f(x)$ ; $\lim\limits_{x\to -\frac{1}{2}^-} f(x)$ ; $\lim\limits_{x\to \pm\infty} f(x)$

- $\lim\limits_{x\to -\frac{1}{2}^+} f(x)$. Tome $\delta>0$. Escreva

$$x = -\frac{1}{2}+\delta .$$



Assim:

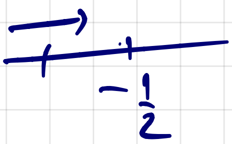
$$\lim_{x\to -\frac{1}{2}^+} \frac{x-1}{1+2x} = \lim_{\delta\to 0^+} \frac{-\frac{1}{2}+\delta-1}{1+2\cdot\left(-\frac{1}{2}+\delta\right)}$$

$$= \lim_{\delta\to 0^+} \frac{-\frac{3}{2}+\delta}{\cancel{1}-\cancel{1}+2\delta} = \lim_{\delta\to 0^+} \frac{-\frac{3}{2}+\delta}{2\delta}$$

$$= \frac{-\frac{3}{2}+0^+}{0^+} = -\infty$$

- $\lim\limits_{x\to -\frac{1}{2}^-} f(x)$. Tome $\delta>0$.

Escreva $x = -\frac{1}{2}-\delta$. Assim:



$$\lim_{x\to -\frac{1}{2}^-} \frac{x-1}{1+2x} = \lim_{\delta\to 0^+} \frac{-\frac{1}{2}-\delta-1}{1+2\cdot\left(-\frac{1}{2}-\delta\right)} =$$

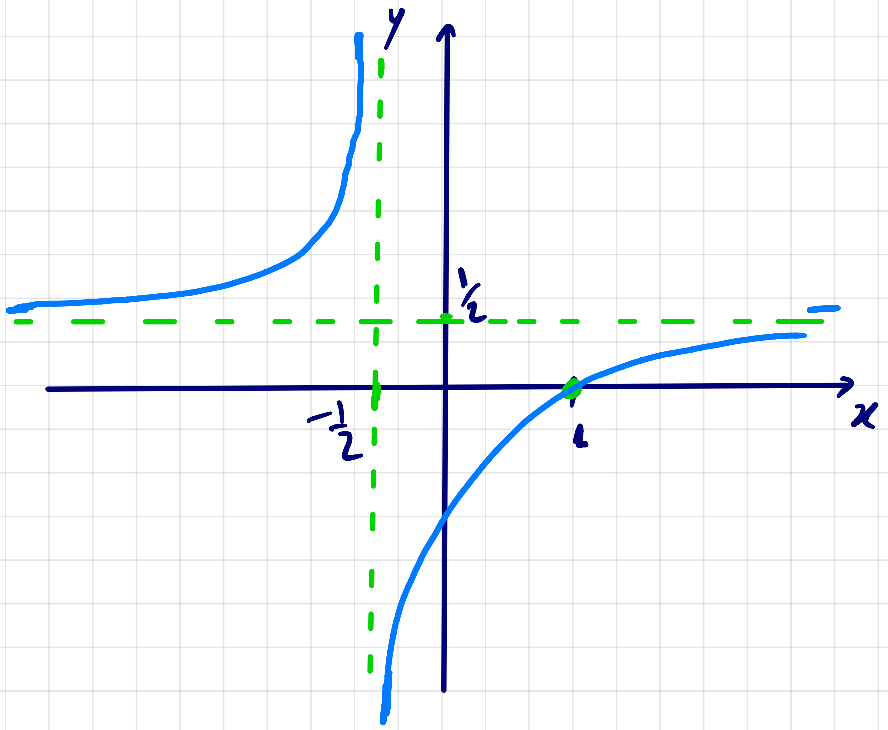
$$= \lim_{\delta \to 0^+} \frac{-\frac{3}{2} - \delta}{\cancel{1} - \cancel{1} - 2\delta} = \lim_{\delta \to 0^+} \frac{-\frac{3}{2} - \delta}{-2\delta}$$

$$= \lim_{\delta \to 0^+} \frac{\overbrace{+\frac{3}{2} + \delta}^{\to 0}}{\underbrace{2\delta}_{\to 0^+}} = +\infty$$

- $$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{x - 1}{1 + 2x} = \lim_{x \to \pm\infty} \frac{\cancel{x}}{2\cancel{x}} = \frac{1}{2}$$

ESBOÇO GRÁFICO DE $f$:

$f(x) \to \frac{1}{2}$ ( ASSINTOTA HORIZONTAL)
$x \to \pm\infty$



(b) $f(x) = \frac{x}{x^2-1}$

$D(f) = ?$ $\qquad x^2-1 \neq 0 \Leftrightarrow x^2 \neq 1 \Leftrightarrow x \neq \pm 1.$

$D(f) = \mathbb{R} \setminus \{-1, 1\}.$

zeros: $f(x) = 0 \Leftrightarrow \frac{x}{x^2-1} = 0 \Leftrightarrow \boxed{x = 0}$

Vamos ter que estudar:

$\lim_{x \to \pm\infty} f(x)$ ; $\lim_{x \to 1^-} f(x)$ ; $\lim_{x \to 1^+} f(x)$ ; $\lim_{x \to -1^-} f(x)$ ; $\lim_{x \to -1^+} f(x)$

- $\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{x}{x^2-1} = \lim_{x \to \pm\infty} \frac{\cancel{x}}{x^{\cancel{2}}} =$

$$= \lim_{x \to \pm\infty} \frac{1}{x} = \frac{1}{\pm\infty} = 0$$

- $\lim_{x \to 1^+} f(x) = ?$ $\qquad$ Tome $\delta > 0$ e escreva $x = 1 + \delta$. Assim:

$$\lim_{x \to 1^+} \frac{x}{x^2-1} = \lim_{x \to 1^+} \frac{x}{(x+1)(x-1)} =$$

$$= \lim_{\delta \to 0^+} \frac{1+\delta}{(1+\delta+1)\cdot(\cancel{1}+\delta-\cancel{1})}$$

$$= \lim_{\delta \to 0^+} \frac{1+\delta}{(2+\delta)\cdot(\delta)} = \frac{1}{2\cdot \underbrace{0^+}_{}} = +\infty$$

$$\underbrace{2\cdot 0^+}_{0^+}$$

- $\lim_{x \to 1^-} f(x) = ?$  Fisso $\delta > 0$, e scrivo $x = 1 - \delta$. Avremo:

$\to 1$

$$\lim_{x \to 1^-} \frac{x}{(x+1)(x-1)} = \lim_{\delta \to 0^+} \frac{1-\delta}{(1-\delta+1)(\cancel{1}-\delta-\cancel{1})}$$

$$= \lim_{\delta \to 0^+} \frac{1-\delta}{(2-\delta)\cdot(-\delta)} = \frac{1}{\underbrace{2\cdot(-0^+)}_{0^-}} = -\infty$$

- $\lim_{x \to -1^+} f(x) = ?$  Fisso $\delta > 0$ e scrivo $x = -1 + \delta$.

$-1 \quad -1+\delta$

Avremo:

$$\lim_{x \to -1^+} \frac{x}{(x+1)(x-1)} = \lim_{\delta \to 0^+} \frac{-1+\delta}{(\cancel{-1}+\delta+\cancel{1})(-1+\delta-1)}$$

$$= \lim_{\delta \to 0^+} \frac{-1+\delta}{\delta\cdot(-2+\delta)} = \frac{-1}{\underbrace{0^+\cdot(-2)}_{0^-}} = +\infty$$

- $\lim_{x \to -1^-} f(x) = ?$ Tome $\delta > 0$ e escreva $x = -1 - \delta$.

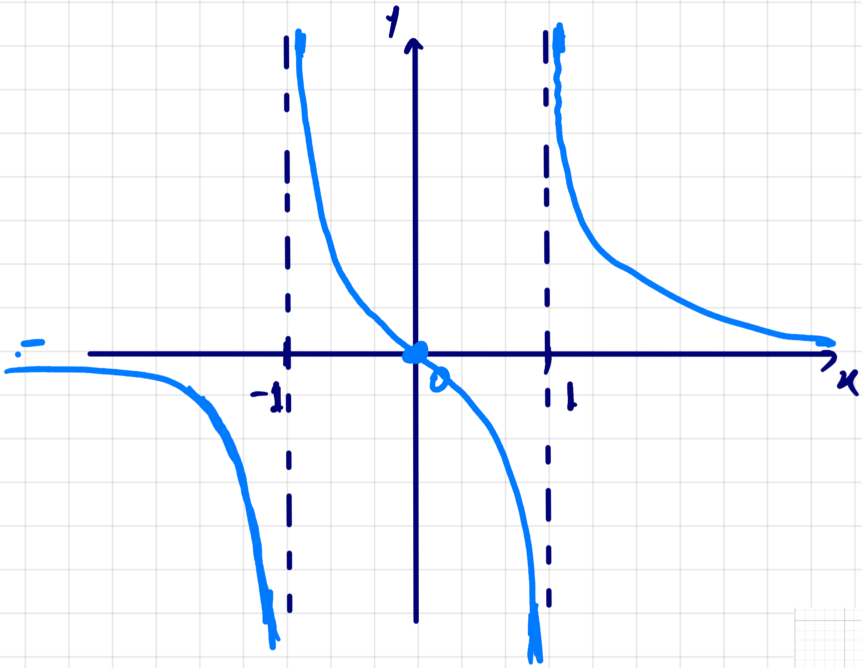
$$\xrightarrow{\quad}\ (\ \ |_{-1}$$

Assim:

$$\lim_{x\to -1^-} \frac{x}{(x+1)(x-1)} = \lim_{\delta \to 0^+} \frac{-1-\delta}{(\cancel{-1}-\delta+\cancel{1})(-1-\delta-1)} =$$

$$= \lim_{\delta\to 0^+} \frac{-1-\delta}{-\delta\cdot(-2-\delta)} = \frac{-1}{\underbrace{-0^+(-2)}_{0^+}} = -\infty$$

ESBOÇO GRÁFICO DA f:




"APUD" GEOGEBRA →