CÁLCULO 1, T1

29/06/24 — AULA 12

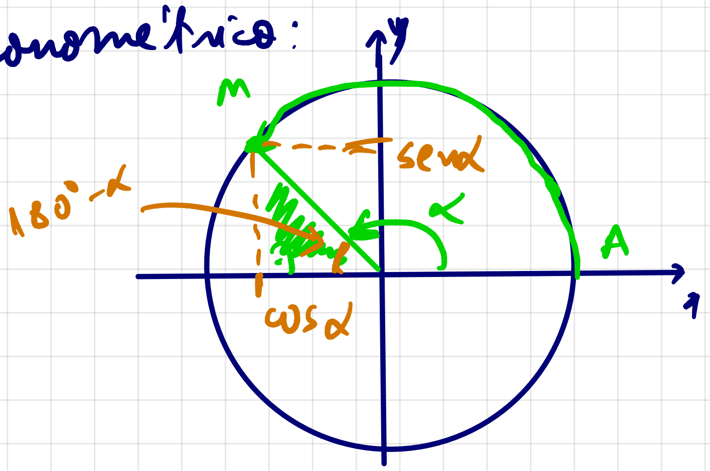
Nesta aula (à distância), apresentaremos importantes fórmulas de Trigonometria (adição e subtração de arcos, arcos duplos e arcos metade).

Antes, porém, apresentaremos mais alguns fatos importantes.

Primeiramente, a relação trigonométrica fundamental, outrora provada para $0 \leq \alpha \leq 90°$, é:

$$\text{sen}^2\alpha + \cos^2\alpha = 1.$$

Note que vale $\forall\, \alpha \in \mathbb{R}$. De fato, basta observar no ciclo trigonométrico:



No caso ilustrado acima, $\alpha \in 2°q$; e então $\text{sen}\,\alpha = \text{sen}(\underbrace{180°-\alpha}_{\in 1°q})$; $\cos\alpha = -\cos(\underbrace{180°-\alpha}_{\in 1°q})$

E então

$$\underbrace{\text{sen}^2(180°-\alpha)}_{\text{sen}^2\alpha} + \underbrace{\cos^2(180°-\alpha)}_{\cos^2\alpha} = 1$$

$$\Rightarrow \text{sen}^2\alpha + \cos^2\alpha = 1, \quad \forall\alpha.$$

Além disso, dado $\text{sen}^2\alpha + \cos^2\alpha = 1$, dividindo por $\cos^2\alpha$ (desde que seja $\neq 0$), vem:

$$\frac{\operatorname{sen}^2\alpha}{\cos^2\alpha} + \cancel{\frac{\cos^2\alpha}{\cos^2\alpha}}_{1} = \frac{1}{\cos^2\alpha}$$

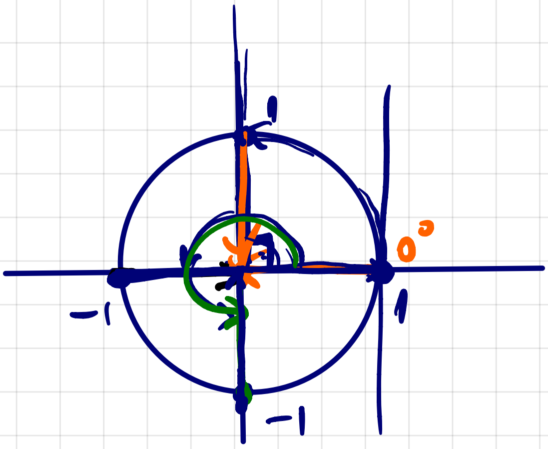
$$\tan^2\alpha + 1 = \sec^2\alpha \implies \boxed{1+\tan^2\alpha = \sec^2\alpha}$$

Dividindo (*) por $\operatorname{sen}^2\alpha \neq 0$, obtemos

$$\cancel{\frac{\operatorname{sen}^2\alpha}{\operatorname{sen}^2\alpha}} + \frac{\cos^2\alpha}{\operatorname{sen}^2\alpha} = \frac{1}{\operatorname{sen}^2\alpha}$$

$$\boxed{1 + \cot^2\alpha = \csc^2\alpha}$$

Valores nos extremidades do ciclo:



$\operatorname{sen} 0° = 0 = \operatorname{sen} 360°$

$\cos 0° = 1 = \cos 360°$

$\tan 0° = 0 = \tan 360°$

$\operatorname{sen} 90° = 1$

$\cos 90° = 0$

$\tan 90° = \nexists \quad \left(\tan 90° = \frac{\operatorname{sen} 90°}{\cos 90°} = \frac{1}{0} = \nexists \right)$

$\operatorname{sen} 180° = 0$

$\cos 180° = -1$

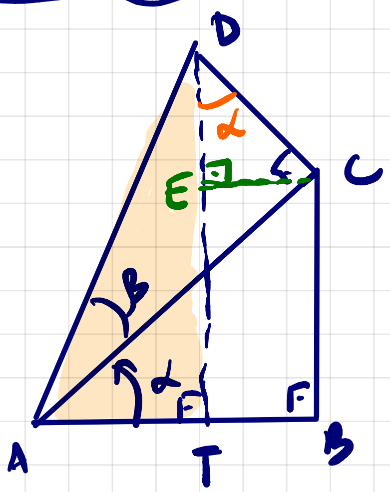
$\tan 180° = 0 \quad \left(\tan 180° = \frac{\operatorname{sen} 180°}{\cos 180°} = \frac{0}{-1} = 0 \right)$

$\text{sen}\,270° = -1$

$\cos 270° = 0$

$\tan 270° = \nexists$ $\quad \left(\tan 270° = \dfrac{\text{sen}\,270°}{\cos 270°} = \dfrac{-1}{0} = \nexists \right)$

ADIÇÃO E SUBTRAÇÃO DE ARCOS:



$\text{sen}(\alpha + \beta) = \,?$

obs.: por simplicidade, no esquema ao lado temos $\alpha, \beta \in 1°q$; mas a fórmula a ser deduzida valerá $\forall \alpha, \beta \in \mathbb{R}$.

$$\text{sen}(\alpha+\beta) = \frac{\overline{DT}}{\overline{AD}} = \frac{\overline{DE} + \overline{ET}}{\overline{AD}} = \frac{\overline{DE} + \overline{BC}}{\overline{AD}} =$$

pois $\overline{ET} = \overline{BC}$

$$= \frac{\overline{DE}}{\overline{AD}} + \frac{\overline{BC}}{\overline{AD}} = \frac{\overline{DE}}{\overline{AD}} \cdot \frac{\overline{DC}}{\overline{DC}} + \frac{\overline{BC}}{\overline{AD}} \cdot \frac{\overline{AC}}{\overline{AC}}$$

$$= \frac{\overline{DE}}{\overline{DC}} \cdot \frac{\overline{DC}}{\overline{AD}} + \frac{\overline{BC}}{\overline{AC}} \cdot \frac{\overline{AC}}{\overline{AD}}$$

$$= \cos\alpha \cdot \text{sen}\,\beta + \text{sen}\,\alpha \cdot \cos\beta$$

Ou seja, acabamos de mostrar que

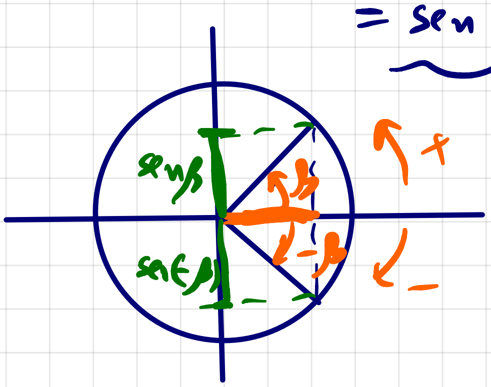
$$\boxed{\operatorname{sen}(\alpha+\beta) = \operatorname{sen}\alpha \cdot \cos\beta + \operatorname{sen}\beta \cdot \cos\alpha.}$$

As demais fórmulas podem ser obtidas da acima.

- $\operatorname{sen}(\alpha-\beta) = \operatorname{sen}(\alpha+(-\beta))$

$$= \operatorname{sen}\alpha \cdot \cos(-\beta) + \operatorname{sen}(-\beta)\cdot\cos\alpha$$

$$= \operatorname{sen}\alpha \cdot \cos\beta - \operatorname{sen}\beta \cdot \cos\alpha$$



$\cos(\beta) = \cos(-\beta)$

$\operatorname{sen}(-\beta) = -\operatorname{sen}\beta$

- $\cos(\alpha+\beta) = \operatorname{sen}(90-(\alpha+\beta))$

$$= \operatorname{sen}[(90^\circ-\alpha)-\beta] =$$

$$= \underbrace{\operatorname{sen}(90^\circ-\alpha)}_{\cos\alpha} \cdot \cos\beta - \operatorname{sen}\beta \cdot \underbrace{\cos(90^\circ-\alpha)}_{\operatorname{sen}\alpha}$$

$$= \cos\alpha \cdot \cos\beta - \operatorname{sen}\alpha \cdot \operatorname{sen}\beta$$

- $\cos(\alpha-\beta) = \cos\alpha \cdot \cos\beta + \operatorname{sen}\alpha \cdot \operatorname{sen}\beta$ (exercício)

- $\tan(\alpha+\beta) = ?$

$$\tan(\alpha+\beta) = \frac{\operatorname{sen}(\alpha+\beta)}{\cos(\alpha+\beta)} =$$

$$= \frac{\operatorname{sen}\alpha\cdot\cos\beta + \operatorname{sen}\beta\cos\alpha}{\cos\alpha\cos\beta - \operatorname{sen}\alpha\operatorname{sen}\beta} = \frac{\frac{\operatorname{sen}\alpha\cancel{\cos\beta}}{\cos\alpha\cancel{\cos\beta}} + \frac{\operatorname{sen}\beta\cancel{\cos\alpha}}{\cancel{\cos\alpha}\cos\beta}}{\frac{\cancel{\cos\alpha\cos\beta}}{\cancel{\cos\alpha\cos\beta}} - \frac{\operatorname{sen}\alpha\cdot\operatorname{sen}\beta}{\cos\alpha\cos\beta}}$$

$$= \frac{\frac{\operatorname{sen}\alpha}{\cos\alpha} + \frac{\operatorname{sen}\beta}{\cos\beta}}{1 - \frac{\operatorname{sen}\alpha}{\cos\alpha}\cdot\frac{\operatorname{sen}\beta}{\cos\beta}} = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\cdot\tan\beta}$$

- $\tan(\alpha-\beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\cdot\tan\beta}$ (exercício)

FÓRMULAS DA PROSTAFÉRESE (TRANSFORMAÇÃO DE SOMA EM PRODUTO)

Exemplo

$$+\begin{cases} a+b = p \\ a-b = q \end{cases}$$

$$2a = p+q \Rightarrow a = \frac{p+q}{2}$$

$$\Rightarrow b = \frac{p-q}{2}$$

$$\operatorname{sen}(a+b) = \operatorname{sen}a\cdot\cos b + \cancel{\operatorname{sen}b\cdot\cos a}$$

$$+\ \operatorname{sen}(a-b) = \operatorname{sen}a\cdot\cos b - \cancel{\operatorname{sen}b\cdot\cos a}$$

$$\operatorname{sen}(\underbrace{a+b}_{p}) + \operatorname{sen}(\underbrace{a-b}_{q}) = 2\cdot\operatorname{sen}\underline{a}\cdot\cos\underline{b}$$

$$\boxed{\operatorname{sen} p + \operatorname{sen} q = 2 \cdot \operatorname{sen} \frac{p+q}{2} \cdot \cos \frac{p-q}{2}}$$

$$\operatorname{sen}(a+b) = \operatorname{sen} a \cdot \cancel{\cos b} + \operatorname{sen} b \cdot \cos a$$

$$- \quad \operatorname{sen}(a-b) = \cancel{\operatorname{sen} a \cos b} - \operatorname{sen} b \cos a$$

$$\underbrace{\operatorname{sen}(a+b)}_{p} - \underbrace{\operatorname{sen}(a-b)}_{q} = +2 \operatorname{sen} b \cdot \cos a$$

$$\Rightarrow \boxed{\operatorname{sen} p - \operatorname{sen} q = 2 \cdot \operatorname{sen} \frac{p-q}{2} \cdot \cos \frac{p+q}{2}}$$

$$+ \begin{cases} \cos(a+b) = \cos a \cdot \cos b - \cancel{\operatorname{sen} a \cdot \operatorname{sen} b} \\ \cos(a-b) = \cos a \cdot \cos b + \cancel{\operatorname{sen} a \cdot \operatorname{sen} b} \end{cases}$$

$$\cos\underbrace{(a+b)}_{p} + \cos\underbrace{(a-b)}_{q} = 2 \cdot \cos a \cdot \cos b$$

$$\boxed{\cos p + \cos q = 2 \cdot \cos \frac{p+q}{2} \cdot \cos \frac{p-q}{2}}$$

$$\boxed{\cos p - \cos q = -2 \cdot \operatorname{sen} \frac{p+q}{2} \cdot \operatorname{sen} \frac{p-q}{2}}$$

EXEMPLOS:

01) Determine $\cos 75°$.

SOLUÇÃO: $\cos 75° = \cos(30° + 45°)$

$= \cos 30° \cdot \cos 45° - \operatorname{sen} 30° \cdot \operatorname{sen} 45°$

$= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} - \frac{1}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{6} - \sqrt{2}}{4}$

02) Dados $x, y \in 1^{\circ}q$ tais que $x + y = 120°$ sabendo que $\operatorname{sen} y = \frac{1}{3}$, determine $\cos x$.

SOLUÇÃO:

$x + y = 120°$

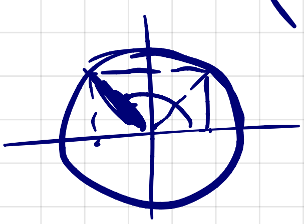
$x = 120° - y$

$\Rightarrow \cos x = \cos(120° - y) =$

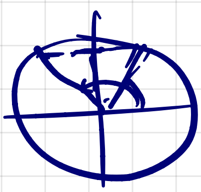
$= \cos 120° \cdot \cos y + \operatorname{sen} 120° \cdot \operatorname{sen} y$

• $\cos 120° = -\cos(180° - 120°) = -\cos 60° = -\frac{1}{2}$

Reduzir ao $1^{\circ}q$



• $\operatorname{sen} 120° = \operatorname{sen}(180° - 120°) = \operatorname{sen} 60° = \frac{\sqrt{3}}{2}$



Daí, temos:

$$\cos x = -\frac{1}{2} \cdot \underbrace{\cos y}_{?} + \frac{\sqrt{3}}{2} \cdot \frac{1}{3}$$

$$\operatorname{sen}^2 y + \cos^2 y = 1$$

$$\cos^2 y = 1 - \operatorname{sen}^2 y$$

$$\cos y = + \sqrt{1 - \operatorname{sen}^2 y} \qquad (y \in 1^{\circ}q)$$

$$\cos y = + \sqrt{1 - \left(\frac{1}{3}\right)^2}$$

$$= \sqrt{1 - \frac{1}{9}} = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}$$

Portanto;

$$\cos x = -\frac{1}{2}\left(\frac{2\sqrt{2}}{3}\right) + \frac{\sqrt{3}}{2} \cdot \frac{1}{3} = \frac{-2\sqrt{2}}{6} + \frac{\sqrt{3}}{6}$$

$$= \frac{\sqrt{3} - 2\sqrt{2}}{6}$$

Iniciamos estudos sobre trigonometria.
Vimos várias fórmulas, tais como as fórmulas de adição e subtração de arcos. Por exemplo:

$$\text{sen}(a+b) = \text{sen}\,a \cdot \cos b + \text{sen}\,b \cdot \cos a \;;$$

$$\cos(a+b) = \cos a \cdot \cos b - \text{sen}\,a \cdot \text{sen}\,b.$$

Vamos estudar agora as fórmulas do arco duplo e do arco metade.

FÓRMULAS DO ARCO DUPLO.

Basta considerar $2a = a + a$ e usar as fórmulas já provadas para adição de arcos.

- $\text{sen}\,2a = \text{sen}(a+a) = \text{sen}\,a \cdot \cos a + \text{sen}\,a \cdot \cos a$

$$= 2 \cdot \text{sen}\,a \cdot \cos a$$

$$\boxed{\text{sen}\,2a = 2 \cdot \text{sen}\,a \cdot \cos a}$$

• $\cos 2a = \cos(a+a) = \cos a \cdot \cos a - \operatorname{sen} a \cdot \operatorname{sen} a$

$= \cos^2 a - \operatorname{sen}^2 a$

$$\Rightarrow \boxed{\cos 2a = \cos^2 a - \operatorname{sen}^2 a}$$

Obs.: Não confunda com a rel. trigonom. fundamental:

$\operatorname{sen}^2 a + \cos^2 a = 1$

• $\tan 2a = \tan(a+a) = \dfrac{\tan a + \tan a}{1 - \tan a \cdot \tan a}$

$= \dfrac{2 \cdot \tan a}{1 - \tan^2 a}$

$$\Rightarrow \boxed{\tan 2a = \frac{2 \tan a}{1 - \tan^2 a}}$$

FÓRMULAS DO ARCO METADE:

Já temos que $\cos 2a = \cos^2 a - \operatorname{sen}^2 a$

Como $\operatorname{sen}^2 a + \cos^2 a = 1 \Rightarrow \cos^2 a = 1 - \operatorname{sen}^2 a$

$$\cos 2a = (1 - \operatorname{sen}^2 a) - \operatorname{sen}^2 a$$

$$= 1 - 2 \operatorname{sen}^2 a$$

$$\Rightarrow 2 \operatorname{sen}^2 a = 1 - \cos 2a$$

$$\Rightarrow \boxed{\operatorname{sen}^2 a = \frac{1 - \cos 2a}{2}}$$

Escrevendo $2a = x$, teremos:

$$\operatorname{sen}^2 \frac{x}{2} = \frac{1 - \cos x}{2}$$

ou ainda:

$$\operatorname{sen} \frac{x}{2} = \pm \sqrt{\frac{1 - \cos x}{2}}$$

ANALISAR SINAL NO QUADRANTE

Do mesmo modo se mostra que

$$\cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}} \; ; \; e$$

que

$$\underbrace{\tan\frac{x}{2}} = \frac{\operatorname{sen}\frac{x}{2}}{\cos\frac{x}{2}} = \underbrace{\pm\sqrt{\frac{1-\cos x}{1+\cos x}}}.$$

Ex-1 Determine o valor de $\cos 37{,}5^\circ$.

Solução: Note que $37{,}5^\circ = \frac{75^\circ}{2}$.

Assim, escreva $x = 75^\circ$.

Vamos, então determinar $\cos\frac{x}{2}$.

Então,

$$\cos\frac{x}{2} = \cos 37{,}5^\circ = +\sqrt{\frac{1+\cos 75^\circ}{2}} \; ;$$

1º Q

Como $\cos 75^\circ = \cos(30^\circ + 45^\circ) =$

$= \cos 30^\circ . \cos 45^\circ - \operatorname{sen} 30^\circ . \operatorname{sen} 45^\circ$

$= \frac{\sqrt{3}}{2} . \frac{\sqrt{2}}{2} - \frac{1}{2} . \frac{\sqrt{2}}{2} = \frac{\sqrt{6}-\sqrt{2}}{4} = \frac{\sqrt{2}(\sqrt{3}-1)}{4}$

Assim, obtemos:

$$\cos 37{,}5^\circ = \sqrt{\frac{1 + \frac{\sqrt{2}(\sqrt{3}-1)}{4}}{2}} = \cdots = \frac{1}{2}\sqrt{\frac{4+\sqrt{6}-\sqrt{2}}{2}}.$$

02) Se $\operatorname{sen} x + \cos x = \frac{1}{\sqrt{3}}$, calcule $\cos 2x$. $(2x \in 1^{\circ} q)$

Solução: Elevando a igualdade ao quadrado, obtemos:

$$(\operatorname{sen} x + \cos x)^2 = \left(\frac{1}{\sqrt{3}}\right)^2$$

$$\operatorname{sen}^2 x + \underbrace{2 \cdot \operatorname{sen} x \cdot \cos x}_{\operatorname{sen} 2x} + \cos^2 x = \frac{1}{3}$$

$$1 + \operatorname{sen} 2x = \frac{1}{3}$$

$$\operatorname{sen} 2x = \frac{1}{3} - 1$$

$$\operatorname{sen} 2x = -\frac{2}{3}$$

$\operatorname{sen}^2 \alpha + \cos^2 \alpha = 1 \ . \ \forall \alpha.$

$$\operatorname{sen}^2 2x + \cos^2 2x = 1$$

$$\left(-\frac{2}{3}\right)^2 + \cos^2 2x = 1$$

$$\cos^2 2x = 1 - \frac{4}{9} = \frac{5}{9}$$

$$\cos 2x = \pm \frac{\sqrt{5}}{3}$$

QUAL O QUADRANTE?

$$\cos 2x = +\frac{\sqrt{5}}{3}$$

$2x \in 1^{\circ} q.$