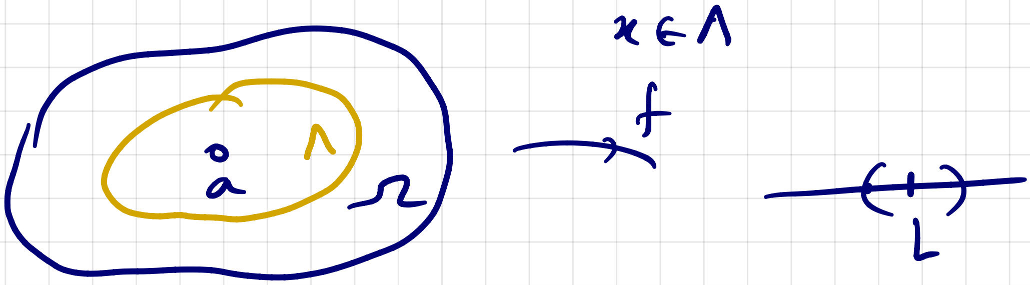
CÁLCULO 3

29/04/24 (AULA 07)

No final da aula passada, no estudo de limites de funções a várias variáveis, vimos o resultado:

PROPOSIÇÃO: Seja $f : \Omega \subset \mathbb{R}^m \to \mathbb{R}$ uma função, $\Lambda \subset \Omega$, $\Lambda \neq \emptyset$ tal que $a \in \Omega' \cap \Lambda'$ (ou seja, $a \in \mathbb{R}^m$ é um ponto de acumulação tanto de Ω quanto de Λ). Então,

$$\lim_{x \to a} f(x) = l \implies \lim_{\substack{x \to a \\ x \in \Lambda}} f(x) = l.$$



COROLÁRIO: Sejam $f : \Omega \subset \mathbb{R}^m \to \mathbb{R}$ uma função, $A, B \subset \Omega$ tais que $a \in \mathbb{R}^m$ é ponto de acumulação dos conjuntos A, B e Ω. Então, se

$$\lim_{\substack{x \to a \\ x \in A}} f(x) \neq \lim_{\substack{x \to a \\ x \in B}} f(x) \implies \nexists \lim_{x \to a} f(x).$$

Obs: O equivalente a este resultado, em $\mathbb{R}$, será:

$$\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x) \implies \nexists \lim_{x \to a} f(x)$$

DEMONSTR.: Por absurdo, suponha que

$$\exists \lim_{x\to a} f(x).$$

Então, sendo $a \in A' \cap B' \cap \Omega'$ (i.e. a é ponto de acumulação dos três conjuntos), segue pela PROP. da aula passada lembrada acima, que

$$\lim_{x\to a} f(x) = \lim_{\substack{x\to a\\ x\in A}} f(x)$$

e

$$\lim_{x\to a} f(x) = \lim_{\substack{x\to a\\ x\in B}} f(x)$$

MAS, por hipótese,

$$\lim_{\substack{x\to a\\ x\in A}} f(x) \neq \lim_{\substack{x\to a\\ x\in B}} f(x),$$

um absurdo!

Portanto, $\nexists \lim_{x\to a} f(x)$

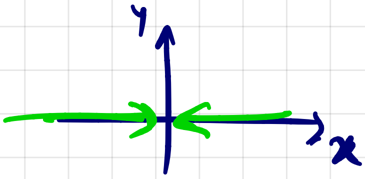
$\square$

Vejamos alguns exemplos:

01) Verifique se $\exists \lim_{(x,y)\to(0,0)} \dfrac{3x^2y}{x^3+y^3}$.

SOLUÇÃO:

Vamos tomar conjuntos de $\mathbb{R}^2$ que tenham $(0,0)$ como ponto de acumulação. Ou seja, deverão ser curvas que passem pela origem do $\mathbb{R}^2$.

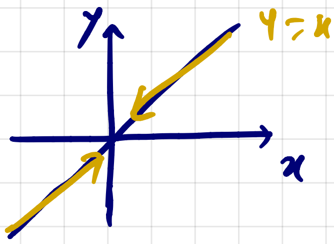
- 

$A_1 = \{(x,y) \in \mathbb{R}^2 : y = 0\}$ (eixo ox)

Então:

$$\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in A_1}} f(x,y) = \lim_{\substack{(x,y)\to(0,0)\\ y=0}} \frac{3x^2y}{x^3+y^3} = \lim_{\substack{(x,y)\to(0,0)\\ y=0}} \frac{3x\cdot(0)^2}{x^3+0^3} =$$

$$= \lim_{\substack{(x,y)\to(0,0)\\ y=0}} \frac{0}{x^3} = 0$$

- 

$A_2 = \{(x,y) \in \mathbb{R}^2 : y = x\}$

$$\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in A_2}} f(x,y) = \lim_{\substack{(x,y)\to(0,0)\\ y=x}} \frac{3x^2y}{x^3+y^3} = \lim_{\substack{(x,y)\to(0,0)\\ y=x}} \frac{3\cdot x^2\cdot x}{x^3+x^3}$$

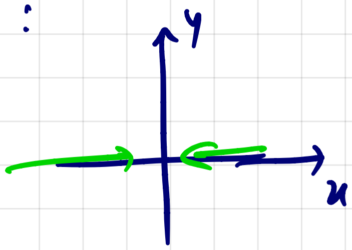
$$= \lim_{\substack{(x,y)\to(0,0)\\ y=x}} \frac{3\cancel{x^3}}{2\cancel{x^3}} = \frac{3}{2}$$

$$\Rightarrow \lim_{\substack{(x,y)\to(0,0)\\ y=0}} f(x,y) = 0 \neq \frac{3}{2} = \lim_{\substack{(x,y)\to(0,0)\\ y=x}} f(x,y)$$

Portanto, $\nexists \lim_{(x,y)\to(0,0)} f(x,y)$.

02) Verifique se $\exists \lim\limits_{(x,y)\to(0,0)} \dfrac{\sqrt{x^2+y^2}}{x^2-y^2}$

Solução:

• 

$$A_1 = \{(x,y) \in \mathbb{R}^2 : y = 0\}$$

$$\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in A_1}} f(x,y) = \lim_{\substack{(x,y)\to(0,0)\\ y=0}} \frac{\sqrt{x^2+y^2}}{x^2-y^2} = \lim_{\substack{(x,y)\to(0,0)\\ y=0}} \frac{\sqrt{x^2}}{x^2} =$$

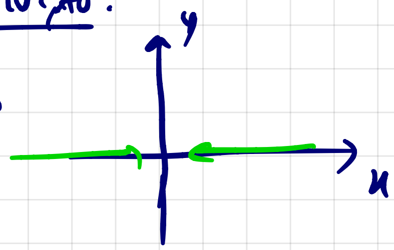
$$= \lim_{\substack{(x,y)\to(0,0)\\ y=0}} \frac{|x|}{x^2} = \begin{cases} \lim\limits_{\substack{(x,y)\to(0,0)\\ y=0,\ x\geq 0}} \dfrac{\cancel{x}}{x^{\cancel{2}}} = +\infty \\ \lim\limits_{\substack{(x,y)\to(0,0)\\ y=0,\ x<0}} -\dfrac{\cancel{x}}{x^{\cancel{2}}} = +\infty \end{cases}$$

$$\Rightarrow \lim_{\substack{(x,y)\to(0,0)\\(x,y)\in A_1}} f(x,y) = +\infty .$$

Logo, $\nexists \lim\limits_{(x,y)\to(0,0)} f(x,y)$, pois por um caminho já resultou em ∞.

03) Verifique se $\exists \lim\limits_{(x,y)\to(0,0)} \dfrac{x\sqrt{y}}{x^2+y^2}$

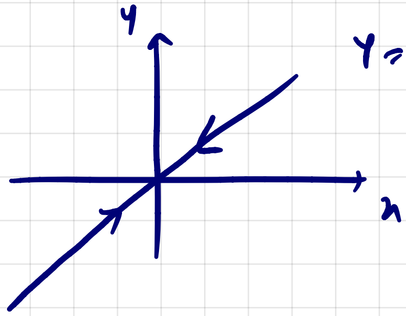
Solução:

• 

$$A_1 = \{(x,y)\in\mathbb{R}^2 : y=0\}$$

Então:

$$\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in A_1}} f(x,y) = \lim_{\substack{(x,y)\to(0,0)\\ y=0}} \frac{x\sqrt{y}}{x^2+y^2} = \lim_{\substack{(x,y)\to(0,0)\\ y=0}} \frac{0}{x^2} = 0.$$

• 

$$A_2 = \{(x,y)\in\mathbb{R}^2 : y=x\}.$$

$$\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in A_2}} f(x,y) = \lim_{\substack{(x,y)\to(0,0)\\ y=x}} \frac{x\sqrt{y}}{x^2+y^2} = \lim_{\substack{(x,y)\to(0,0)\\ y=x}} \frac{\cancel{x}\sqrt{x}}{2x^{\cancel{2}}} =$$

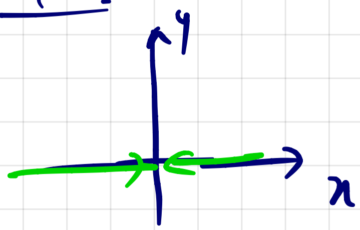
$$\lim_{\substack{(x,y)\to(0,0)\\ y=x}} \frac{\sqrt{x}}{2x} = \lim_{\substack{(x,y)\to(0,0)\\ y=x}} \frac{\sqrt{x}}{2x}\cdot\frac{\sqrt{x}}{\sqrt{x}} = \lim_{\substack{(x,y)\to(0,0)\\ y=x}} \frac{\cancel{x}}{2\cancel{x}\sqrt{x}} =$$

$$= \lim_{\substack{(x,y)\to(0,0)\\ y=x}} \frac{1}{2\sqrt{x}} = \nexists$$

Portanto, $\nexists \lim\limits_{(x,y)\to(0,0)} f(x,y)$

04) Verifique se $\exists \lim\limits_{(x,y)\to(0,0)} \dfrac{x^2+y^2}{\sqrt{x^2+y^2}}$.

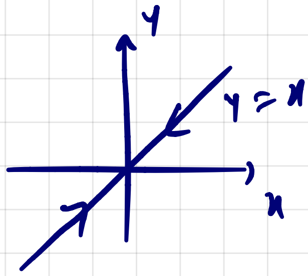
Solução:

- 

$A_1 = \{(x,y)\in\mathbb{R}^2 : y=0\}$.

$$\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in A_1}} f(x,y) = \lim_{\substack{(x,y)\to(0,0)\\y=0}} \frac{x^2+y^2}{\sqrt{x^2+y^2}} = \lim_{\substack{(x,y)\to(0,0)\\y=0}} \frac{x^2}{\sqrt{x^2}} =$$

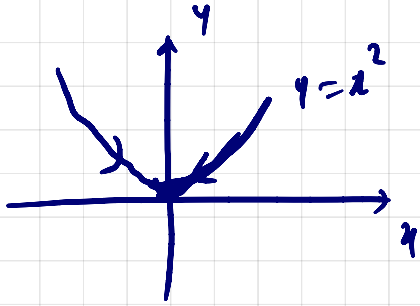
$$= \lim_{\substack{(x,y)\to(0,0)\\y=0}} \frac{x^2}{|x|} = \begin{cases} \lim\limits_{\substack{(x,y)\to(0,0)\\y=0,\ x\geq 0}} \dfrac{x^{\cancel{2}}}{\cancel{x}} = 0 \\ \lim\limits_{\substack{(x,y)\to(0,0)\\y=0,\ x<0}} \dfrac{x^{\cancel{2}}}{-\cancel{x}} = 0 \end{cases} = 0 \;//$$

- 

$A_2 = \{(x,y)\in\mathbb{R}^2 : y=x\}$.

$$\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in A_2}} f(x,y) = \lim_{\substack{(x,y)\to(0,0)\\y=x}} \frac{x^2+y^2}{\sqrt{x^2+y^2}} = \lim_{\substack{(x,y)\to(0,0)\\y=x}} \frac{2x^2}{\sqrt{2x^2}} =$$

$$\lim_{\substack{(x,y)\to(0,0)\\y=x}} \frac{2x^2}{\sqrt{2}\cdot|x|} = \begin{cases} \lim\limits_{\substack{(x,y)\to(0,0)\\y=x,\ x\geq 0}} \dfrac{2x^{\cancel{2}}}{\sqrt{2}\,\cancel{x}} = 0 \\ \lim\limits_{\substack{(x,y)\to(0,0)\\y=x,\ x<0}} \dfrac{2x^{\cancel{2}}}{-\sqrt{2}\,\cancel{x}} = 0 \end{cases} = 0$$

• 

$$\Lambda_3 = \{(x,y)\in\mathbb{R}^2 : y = x^2\}$$

$$\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in\Lambda_3}} f(x,y) = \lim_{\substack{(x,y)\to(0,0)\\y=x^2}} \frac{x^2+y^2}{\sqrt{x^2+y^2}} = \lim_{\substack{(x,y)\to(0,0)\\y=x^2}} \frac{x^2+x^4}{\sqrt{x^2+x^4}} =$$

$$= \lim_{\substack{(x,y)\to(0,0)\\y=x^2}} \frac{x^2(1+x^2)}{|x|\cdot\sqrt{1+x^2}} = \begin{cases} \lim\limits_{\substack{(x,y)\to(0,0)\\y=x^2,\ x\geq 0}} \dfrac{\overset{0}{x^{\cancel{2}}}\overset{1}{(1+x^2)}}{\cancel{x}\underset{1}{\sqrt{1+x^2}}} = 0 \\[2ex] \lim\limits_{\substack{(x,y)\to(0,0)\\y=x^2,\ x<0}} \dfrac{\overset{0}{x^{\cancel{2}}}\overset{1}{(1+x^2)}}{-\cancel{x}\cdot\underset{1}{\sqrt{1+x^2}}} = 0 \end{cases}$$

$$= 0$$

Ou seja, pelos três "caminhos" encontramos que o limite é zero, o que "sugere" que o limite existe e deve ser igual a zero.

AF: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2}} = 0$

Dado $\varepsilon>0$, precisamos encontrar $\delta>0$, tal que, $\forall (x,y)\in\mathbb{R}^2$: $0<d_2((x,y);(0,0))<\delta$ implique em $|f(x,y)-0|<\varepsilon$.

$$0<\sqrt{(x-0)^2+(y-0)^2}<\delta$$

Ou seja; devemos ter $|f(x,y)-0|<\varepsilon$, sempre que $0<\underline{\sqrt{x^2+y^2}<\delta}$.

Analisando $|f(x,y)-0|$:

$$|f(x,y)-0| = \left|\frac{x^2+y^2}{\sqrt{x^2+y^2}}-0\right| = \frac{x^2+y^2}{\sqrt{x^2+y^2}} =$$

$$= \frac{x^2+y^2}{\sqrt{x^2+y^2}} \times \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} = \frac{\cancel{(x^2+y^2)}\cdot\sqrt{x^2+y^2}}{\cancel{x^2+y^2}} =$$

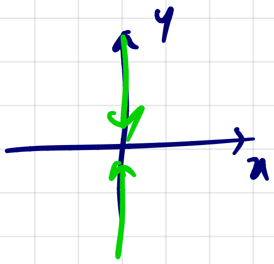
$$= \underbrace{\sqrt{x^2+y^2}}_{<\delta} < \delta = \varepsilon .$$

Ou seja, basta tomar $\delta=\varepsilon$.

$\square$

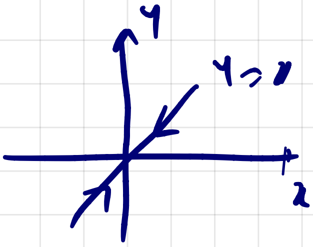
05) Verifique se $\exists \lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^2+y^2}$.

Solução:



- $A_1 = \{(x,y)\in\mathbb{R}^2 : x=0\}$.

$$\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in A_1}} f(x,y) = \lim_{\substack{(x,y)\to(0,0)\\x=0}} \frac{x^2y}{x^2+y^2} = \lim_{\substack{(x,y)\to(0,0)\\x=0}} \frac{0}{y^2} = 0$$

• 

$\Lambda_2 = \{(x,y) \in \mathbb{R}^2 : y = x\}$

$$\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in\Lambda_2}} f(x,y) = \lim_{\substack{(x,y)\to(0,0)\\y=x}} \frac{x^2y}{x^2+y^2} = \lim_{\substack{(x,y)\to(0,0)\\y=x}} \frac{x^3}{x^2+x^2} =$$

$$= \lim_{\substack{(x,y)\to(0,0)\\y=x}} \frac{x^{\cancel{3}}}{2\cancel{x^2}} = 0$$

Se dois caminhos diferentes resultam mesmo limite. Isto "sugere" que o limite existe.

AF.: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2} = 0.$

Dado $\varepsilon > 0$, precisamos achar $\delta > 0$ tal que,
$\forall (x,y) \in \mathbb{R}^2$: $\quad 0 < d_E((x,y),(0,0)) < \delta \Rightarrow |f(x,y) - 0| < \varepsilon.$

ou seja, $\quad 0 < \sqrt{x^2+y^2} < \delta \Rightarrow |f(x,y)| < \varepsilon.$

Avaliando $|f(x,y)|$:

$$|f(x,y)| = \left|\frac{x^2y}{x^2+y^2}\right| = \frac{x^2 \cdot |y|}{x^2+y^2}.$$

Note que:

$$|y| = \sqrt{y^2} \le \sqrt{y^2+x^2} = \sqrt{x^2+y^2};$$

e

$$x^2 \le x^2+y^2$$

Assim:

$$|f(x,y)| = \frac{x^2\,|y|}{x^2+y^2} \le \frac{x^2 \cdot \sqrt{x^2+y^2}}{x^2+y^2} \le$$

$$\le \frac{\cancel{(x^2+y^2)}\sqrt{x^2+y^2}}{\cancel{x^2+y^2}} = \sqrt{x^2+y^2} < \delta = \varepsilon.$$

Ou seja, basta tomar $\delta = \varepsilon$.

$\square$

TEOREMA (TEOREMA DO SANDUÍCHE) Sejam $f, g, h : \Omega \subset \mathbb{R}^m \to \mathbb{R}$ funções, com $a \in \mathbb{R}^m$ ponto de acumulação de Ω, tais que $f(x) \le g(x) \le h(x)$, $\forall x \in \Omega$.

Se $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$, então

$$\lim_{x\to a} g(x) = L.$$