GABARITO DA TERCEIRA PROVA DE CÁLCULO 1

PROVA I

01) $s(t) = 5t + 20.(t+1)^{-1}$.

- $v(t) = s'(t) = 5 - 20.(t+1)^{-2}.1 = 5 - \frac{20}{(t+1)^2}$

$v(1) = 5 - \frac{20}{(2)^2} = 5 - 5 = 0\ m/s$.

- $a(t) = v'(t) = 0 + 40(t+1)^{-3}.1 = \frac{40}{(t+1)^3}$

$\Rightarrow a(1) = \frac{40}{(2)^3} = 5\ m/s^2$.

02)

(a) $y = (1+x).\text{arctan}\, x^{\frac{1}{2}} = u.v$

$y' = u.v' + u'.v$

$$\left[\begin{array}{l} u = 1+x \Rightarrow u' = 1 \\ v = \text{arctan}\, x^{\frac{1}{2}} \Rightarrow v' = \dfrac{\frac{1}{2}x^{-\frac{1}{2}}}{1+(x^{\frac{1}{2}})^{\cancel{2}}} \end{array}\right.$$

Disso, segue que

$$y' = (1+\cancel{x}) . \frac{\frac{1}{2\sqrt{x}}}{1+\cancel{x}} + 1 . \text{arctan}\sqrt{x}$$

$$\boxed{y' = \frac{1}{2\sqrt{x}} + \text{arctan}\sqrt{x}}$$

(b) $y = \ln\left((1+x^2)^{\frac{1}{2}} - x \right)$

$$y' = \frac{\cancel{\frac{1}{2}}(1+x^2)^{-\frac{1}{2}} \cdot \cancel{2}x - 1}{\sqrt{1+x^2} - x} = \frac{\frac{1}{\sqrt{1+x^2}}\,x - 1}{\sqrt{1+x^2} - x}$$

$$= \frac{\frac{x - \sqrt{1+x^2}}{\sqrt{1+x^2}}}{\sqrt{1+x^2} - x} = - \frac{\cancel{\sqrt{1+x^2} - x}}{\sqrt{1+x^2}} \cdot \frac{1}{\cancel{\sqrt{1+x^2} - x}}$$

$$= - \frac{1}{\sqrt{1+x^2}}$$

(c) $y = \frac{x^2 - x}{\sqrt{x}} = \frac{u}{v}$

$$\left[\begin{array}{l} u = x^2 - x \Rightarrow u' = 2x - 1 \\ v = x^{\frac{1}{2}} \Rightarrow v' = \frac{1}{2} x^{-\frac{1}{2}} \cdot 1 \\ \qquad\qquad\qquad = \frac{1}{2\sqrt{x}} \end{array}\right.$$

$$y' = \frac{v \cdot u' - u \cdot v'}{v^2}$$

$$\Rightarrow y' = \frac{\sqrt{x} \cdot (2x-1) - (x^2 - x) \cdot \frac{1}{2\sqrt{x}}}{(\cancel{\sqrt{x}})^{\cancel{2}}}$$

$$y' = \frac{\frac{2x(2x-1) - x^2 + x}{2\sqrt{x}}}{x} = \frac{4x^2 - 2x - x^2 + x}{2\sqrt{x}} \cdot \frac{1}{x}$$

$$y' = \frac{3x^2 - \cancel{x}}{2x\sqrt{x}} = \frac{3x - 1}{2\sqrt{x}}$$

(d) $y = x^2 \cdot e^{\tan 2x} = u \cdot v$

$\Rightarrow y' = u \cdot v' + u' \cdot v$; onde:

$$\begin{cases} u = x^2 \Rightarrow u' = 2x \\ v = e^{\tan 2x} \Rightarrow v' = e^{\tan 2x} \cdot \sec^2 2x \cdot 2 \end{cases}$$

Logo:

$$y' = x^2 \cdot 2 \cdot \sec^2 2x \cdot e^{\tan 2x} + 2x \cdot e^{\tan 2x}$$

$$y' = 2x \cdot e^{\tan 2x} \left(x^2 \sec^2 2x + 1 \right)$$

03) $x^2 + y^2 + \ln(xy) = \cos y$. Derivando implicitamente em x, obtemos:

$$2x + 2y \cdot y' + \frac{x \cdot y' + 1 \cdot y}{xy} = -\operatorname{sen} y \cdot y'$$

$$2x + 2y \cdot y' + \frac{y'}{y} + \frac{1}{x} = -\operatorname{sen} y \cdot y'$$

$$\left(2y + \frac{1}{y} + \operatorname{sen} y\right) \cdot y' = -\frac{1}{x} - 2x$$

$$\Rightarrow \boxed{y' = \frac{-\frac{1}{x} - 2x}{2y + \frac{1}{y} + \operatorname{sen} y}}$$

04) $x = t^{1/3}$; $y = t^{2/3}$

$$\Rightarrow \frac{dx}{dt} = \frac{1}{3} t^{\frac{1}{3}-1} \cdot 1 \quad ; \quad \frac{dy}{dt} = \frac{2}{3} \cdot t^{\frac{2}{3}-1} \cdot 1$$

$$\Rightarrow \frac{dx}{dt} = \frac{1}{3 \cdot t^{2/3}} \quad ; \quad \frac{dy}{dt} = \frac{2}{3 \cdot t^{1/3}}$$

Assim:

$$y' = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{2}{3 t^{1/3}}}{\frac{1}{3 t^{2/3}}} = \frac{2}{\cancel{3} t^{1/3}} \cdot \frac{\cancel{3} t^{2/3}}{1}$$

$$y' = 2 t^{\frac{2}{3} - \frac{1}{3}} \Rightarrow y' = 2 \underbrace{t^{1/3}}_{= x}$$

$$\Rightarrow \boxed{y' = 2x}$$

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05) $f: \mathbb{R} \to \mathbb{R}$; $f(x) = 2 + (x-3)^{1/3}$.

(a) zeros: $f(x) = 0 \Leftrightarrow 2 + (x-3)^{1/3} = 0$

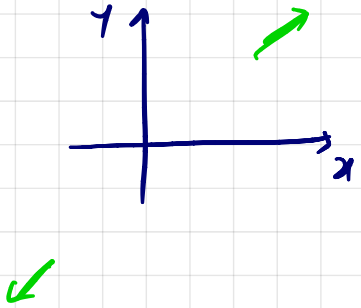
$$\Leftrightarrow (x-3)^{1/3} = -2$$

$$\Leftrightarrow \left((x-3)^{1/3}\right)^3 = (-2)^3$$

$$\Leftrightarrow x - 3 = -8 \Leftrightarrow \boxed{x = -5}$$

Não há assíntotas, pois $f(x) = 2 + \sqrt[3]{x-3}$ está definida $\forall x \in \mathbb{R}$. Além disso,

$$\lim_{x \to \pm\infty} 2 + \sqrt[3]{x-3} = \pm\infty.$$



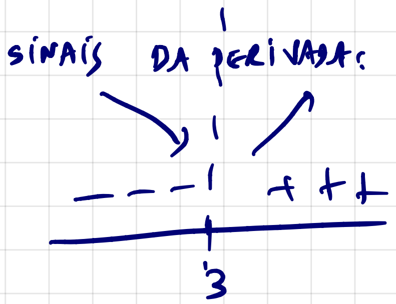
(b) PONTOS CRÍTICOS: onde $f'(x) = 0$ e onde $\nexists\, f'(x)$.

$$f'(x) = 0 + \frac{1}{3}(x-3)^{\frac{1}{3}-1} \cdot 1 \Rightarrow f'(x) = \frac{1}{3\sqrt[3]{(x-3)^2}}$$

Disso:

- $f'(x) = 0 \Leftrightarrow 1 = 0$ (Absurdo). Logo, $\nexists\, x$ tal que $f'(x) = 0$
- $\nexists\, f'(x) \Leftrightarrow x - 3 = 0 \Leftrightarrow \boxed{x = 3}$ ÚNICO PONTO CRÍTICO.

(divisão por zero)



$$x = 0;\quad y' = \frac{1}{3\cdot\sqrt[3]{-3}} < 0$$

$$x = 4;\quad y' = \frac{1}{3\sqrt[3]{1}} > 0$$

Logo, f é decrescente em $(-\infty, 3)$ e crescente em $(3, +\infty)$

(c) Pelo estudo do sinal da derivada feito acima, concluímos que $\nexists$ MÁXIMO. Também $\nexists$ MÍNIMO, pois embora f decresça até 3 e cresça a partir daí, lembre que $\nexists\, f'(3)$.

(d) concavidade e P.I.. estudo do sinal de f''.

$$f'(x) = \frac{1}{3} \cdot (x-3)^{-2/3} \Rightarrow f''(x) = \frac{1}{3} \cdot \left(-\frac{2}{3}\right) \cdot (x-3)^{-\frac{2}{3}-1} \cdot 1$$

$$\Rightarrow f''(x) = \frac{-2}{9} \cdot (x-3)^{-\frac{5}{3}} = -\frac{2}{9\sqrt[3]{(x-3)^5}}$$

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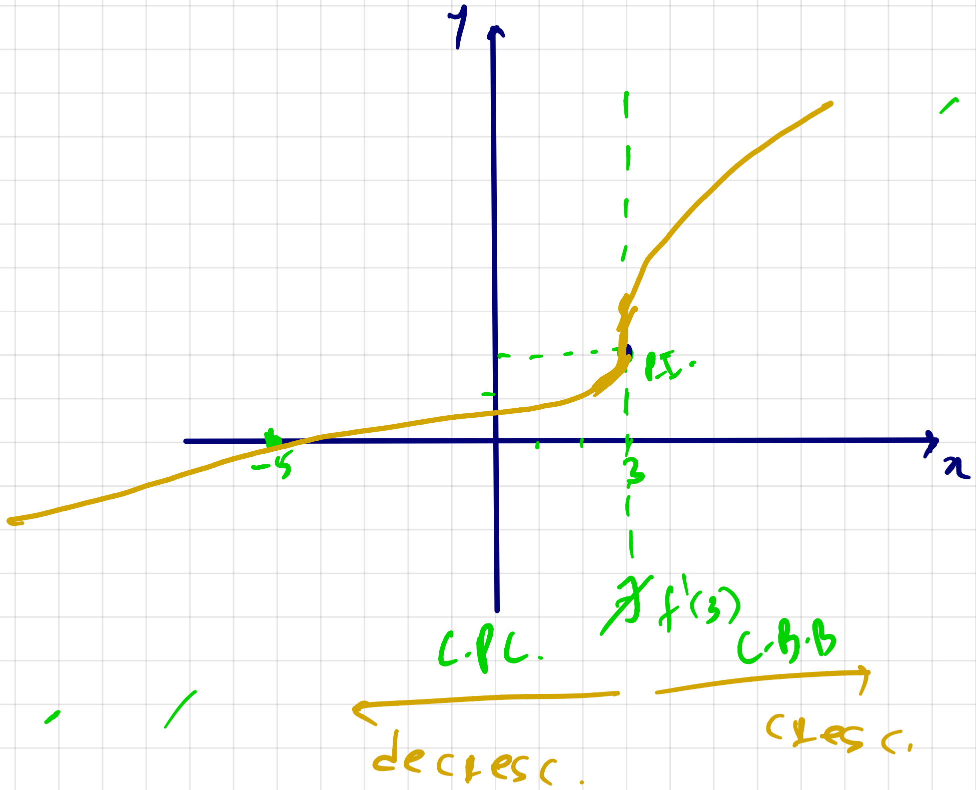
$x = 0$, $f''(x) > 0$

$x = 4$; $f''(x) < 0$

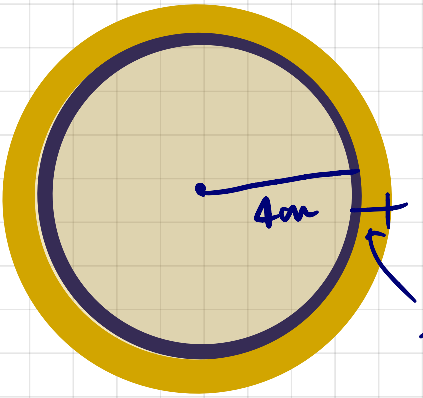
Logo, temos que f é côncava para baixo em $(3, +\infty)$ e côncava para cima em $(-\infty, 3)$. Portanto, $x = 3$ fornece um P.I.

$$P.I. \ (3, f(3)) = (3, 2)$$

(e)



06)



$$\frac{dV}{dt} = -10 \text{ cm}^3/\text{min}.$$

$$\left.\frac{dx}{dt}\right|_{x=2\text{cm}} = ?$$

$$V = \frac{4\pi R^3}{3}, \quad \text{onde } R = 4 + x.$$

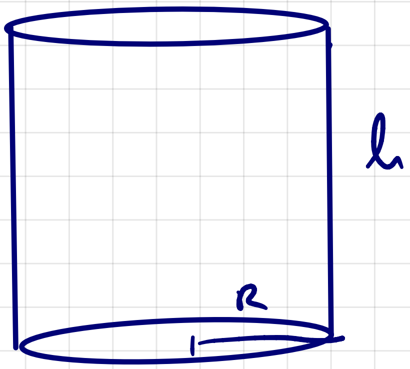
$$\Rightarrow V = \frac{4\pi}{3} \cdot (4+x)^3.$$

$$\frac{dV}{dt} = \frac{4\pi}{\cancel{3}} \cdot \cancel{3} \cdot (4+x)^2 \cdot \frac{dx}{dt}$$

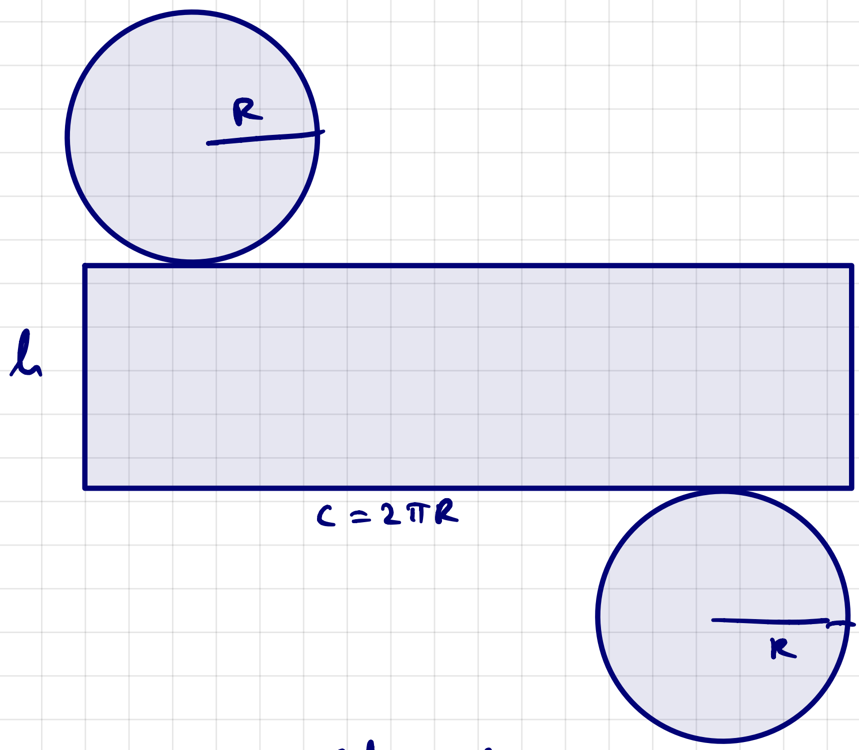
$$\Rightarrow \frac{dx}{dt} = \frac{\frac{dV}{dt}\,)=-10}{4\pi(4+x)^2}. \quad \text{Assim,}$$

$$\left.\frac{dx}{dt}\right|_{x=2\text{cm}} = \frac{-\cancel{10}^{\,5}}{\cancel{4}_{2}\pi \cdot (4+2)^2} = \frac{5}{2\pi \cdot (6)^2} = \frac{5}{72\pi} \text{ cm}/\text{min}$$

07)



LATA ABERTA



$$V = A_b . h$$

$$16\cancel{\pi} = \cancel{\pi} R^2 . h$$

$$\Rightarrow \boxed{h = \frac{16}{R^2}}$$

Vamos obter h e R de tal modo que a área total seja mínima.

$$A = \underbrace{2\pi R . h}_{\text{ÁREA LATERAL}} + 2 . \underbrace{\pi R^2}_{\text{ÁREA DA BASE}}$$

$$A = 2\pi \cancel{R} . \frac{16}{R^{\cancel{2}}} + 2\pi R^2$$

$$A = 32\pi . R^{-1} + 2\pi R^2$$

$$A'(R) = 32\pi . (-R^{-2}) + 2\pi . (2R)$$

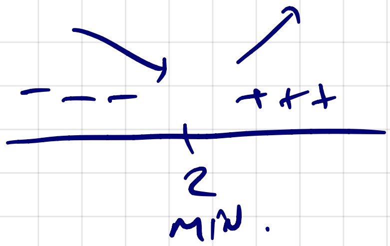
$$A'(R) = -\frac{32\pi}{R^2} + 4\pi R$$

$$A'(R) = 0 \Leftrightarrow -\frac{32\pi}{R^2} + 4\pi R = 0$$

$$\Leftrightarrow \frac{32\cancel{\pi}}{R^2} = 4\cancel{\pi} R$$

$$\Leftrightarrow \quad x^3 = 8 \quad \Leftrightarrow \quad \boxed{R = 2\text{ cm}}$$

SINAL DA DERIVADA:



$$\begin{cases} R = 1; \ A'(R) = -32\pi + 4\pi < 0 \\ R = 10; \ A'(R) = -\dfrac{32\pi}{100} + 400 > 0 \end{cases}$$

Logo, $R = 2\text{cm}$ fornecerá a área mínima.

Neste caso, teremos $h = \left.\dfrac{16}{R^2}\right|_{R=2\text{cm}} = \dfrac{16}{(2)^2} = 4\text{ cm}.$

<u>Resp.:</u> $R = 2\text{ cm}$ e $h = 4\text{cm}$.

08) $f: [1, +\infty) \to \mathbb{R}$, $f(x) = \ln x$.

Dados $x, y \in [1, +\infty)$, com $x < y$ e considere

$$f: [x, y] \to \mathbb{R}, \ f(x) = \ln x.$$

Como f é cont. em $[x, y]$ e derivável em (x, y), pelo T.V.M. segue que $\exists\, c$ entre x e y tal que

$$f'(c) = \frac{f(y) - f(x)}{y - x}$$

$$\Rightarrow \quad f(y) - f(x) = f'(c) \cdot (y - x).$$

Logo: $\quad |f(y) - f(x)| = |f'(c) \cdot (y - x)|$

$$\Rightarrow \quad |f(x) - f(y)| = |f'(c)| \cdot |x - y|, \text{ onde}$$

$$f'(c) = (\ln x)'\Big|_{x=c} = \frac{1}{c}.$$

Como $c \geq 1$, então $\frac{1}{c} \leq 1$; donde segue que

$$|f(x)-f(y)| = \underbrace{|f'(c)|}_{\leq 1} \cdot |x-y| \leq 1 \cdot |x-y|$$

$$\Rightarrow |f(x)-f(y)| \leq 1 \cdot |x-y|, \quad \forall x, y \in (1, +\infty)$$

Portanto, f é de Lipschitz.

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09) Seja $f: (0, +\infty) \to \mathbb{R}$, $f(x) = \sqrt{x} = x^{\frac{1}{2}}$

$$\Rightarrow f'(x) = \frac{1}{2} \cdot x^{-\frac{1}{2}} \cdot 1 \Rightarrow f'(x) = \frac{1}{2\sqrt{x}}.$$

Tome $x = 4$ e $\Delta x = 1$. Então, $x + \Delta x = 4 + 1 = 5$, e disso segue que

$$f(x + \Delta x) \simeq f(x) + f'(x) \cdot \Delta x$$

$$f(5) \simeq f(4) + f'(4) \cdot 1 = \sqrt{4} + \frac{1}{2\sqrt{4}} \cdot 1$$

$$= 2 + \frac{1}{4} = 2 + 0{,}25 = 2{,}25$$

Portanto, obtemos que

$$\sqrt{5} \simeq 2{,}25.$$
