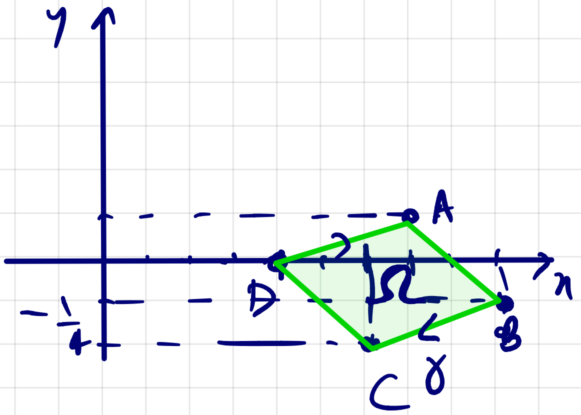
CÁLCULO IV

13/03/24 – AULA 22

AULA DE EXERCÍCIOS:

De uma prova do semestre passado:

Questão 01. [Peso 1,5] Use o Teorema de Green para calcular $\oint_\gamma \cos(x-3y)dx + \ln(x+y)dy$, onde γ é o quadrilátero $ABCD$ de vértices $A(\frac{7}{4},\frac{1}{4})$, $B(\frac{9}{4},-\frac{1}{4})$, $C(\frac{3}{2},-\frac{1}{2})$ e $D(1,0)$.



$$\oint_\gamma P\,dx + Q\,dy = \iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \;(\Longrightarrow)$$

$$P = \cos(x-3y) \Rightarrow \frac{\partial P}{\partial y} = -\operatorname{sen}(x-3y)(-3)$$
$$= 3\operatorname{sen}(x-3y)$$

$$Q = \ln(x+y)$$
$$\Downarrow$$
$$\frac{\partial Q}{\partial x} = \frac{1}{x+y}$$

$$(\Longrightarrow) \iint_\Omega \left(\frac{1}{x+y} - 3\operatorname{sen}(x-3y)\right) dx\,dy$$

Eravamo $\begin{cases} u = x+y \\ v = x-3y \end{cases} \Longrightarrow x = u-y$

$$v = u - y - 3y$$

$$v - u = -4y \Rightarrow \boxed{y = \frac{1}{4}u - \frac{1}{4}v}$$

$$x = u - y$$

$$x = u - \left(\frac{1}{4}u - \frac{1}{4}v\right)$$

$$\boxed{x = \frac{3}{4}u + \frac{1}{4}v}$$

$\quad x \quad y$

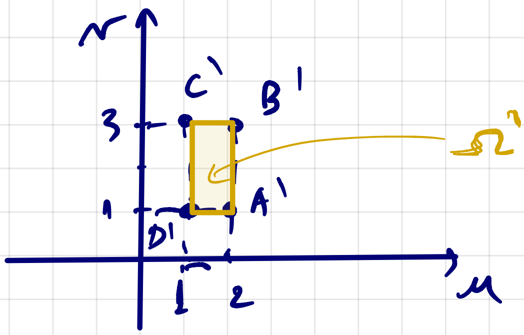
$$(u,v) = (x+y,\ x-3y)$$

$A\left(\frac{7}{4}, \frac{1}{4}\right) \rightsquigarrow A'\left(\frac{7}{4}+\frac{1}{4}, \frac{7}{4}-\frac{3}{4}\right) = (2,1)$

$B\left(\frac{9}{4}, -\frac{1}{4}\right) \rightsquigarrow B'\left(\frac{9}{4}-\frac{1}{4}, \frac{9}{4}+\frac{3}{4}\right) = (2,3)$

$C\left(\frac{3}{2}, -\frac{1}{2}\right) \rightsquigarrow C'\left(\frac{3}{2}-\frac{1}{2}, \frac{3}{2}+\frac{3}{2}\right) = (1,3)$

$D(1,0) \rightsquigarrow D'(1+0, 1+0) = (1,1)$



Além disso:

$$\det J(T)(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} \end{vmatrix} =$$

$$= -\frac{3}{16} - \frac{1}{16} = -\frac{4}{16} = -\frac{1}{4}$$

Assim, temos:

$$\oint_\gamma \cos(x-3y)\,dx + \ln(x+y)\,dy = \iint_\Omega \left(\frac{1}{x+y} - 3\,\mathrm{sen}(x-3y)\right) dx\,dy =$$

$$= \iint_{\Omega'} \left(\frac{1}{u} - 3\,\mathrm{sen}\,v\right) \cdot \Big|\underbrace{\det(T)(u,v)}_{-\frac{1}{4}}\Big| \cdot du\,dv =$$

$$= \int_{v=1}^{v=3}\int_{u=1}^{u=2} \left(\frac{1}{u} - 3\,\mathrm{sen}\,v\right) \cdot \left|-\frac{1}{4}\right| \cdot du\,dv =$$

$$\frac{1}{4}\int_{v=1}^{v=3}\int_{u=1}^{u=2} \frac{du}{u} \cdot dv - \frac{3}{4}\int_{v=1}^{v=3}\int_{u=1}^{u=2} \mathrm{sen}\,v \cdot du\,dv$$

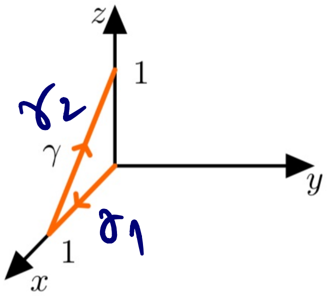
$$= \frac{1}{4}\int_{v=1}^{v=3} dv \cdot \int_{u=1}^{u=2} \frac{du}{u} - \frac{3}{4} \cdot \int_{v=1}^{v=3} \mathrm{sen}\,v\,dv \int_{u=1}^{u=2} du = (\ldots)$$

— TERMINAR. —

PROVA ANTIGA.

Questão 03. Seja o campo vetorial $\vec{F}$ dado por $\vec{F}(x,y,z) = (e^x \cos y, e^x \operatorname{sen} y, z)$.

(a) [Peso 1,0] Calcule $\int_\gamma \vec{F} d\vec{r}$, onde γ é o caminho dado na ilustração abaixo.



(b) [Peso 1,0] Calcule a divergência e o rotacional do campo $\vec{F}$.

$P(x,y,z) = e^x \cdot \cos y$;

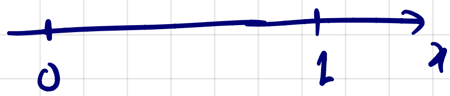
$Q(x,y,z) = e^x \cdot \operatorname{sen} y$

$R(x,y,z) = z$.

(a) $\int_\gamma \vec{F} \cdot d\vec{r}$ $\qquad \gamma = \gamma_1 \cup \gamma_2$

$0 \leq t \leq 1$

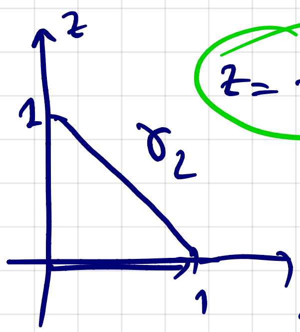
γ_1:



$$\int_\gamma \vec{F} d\vec{r} = \int_{\gamma_1} \vec{F} d\vec{r} + \int_{\gamma_2} \vec{F} d\vec{r}$$

$$\gamma_1(t): \begin{cases} x = t \rightarrow dx = dt \\ y = 0 \rightarrow dy = 0 \\ z = 0 \rightarrow dz = 0 \end{cases}.$$

γ_2:



$z = -x + 1$

PARAMETRIZAÇÃO:

$$\begin{cases} x = t \rightarrow dx = dt \\ y = 0 \rightarrow dy = 0 \\ z = 1 - t \rightarrow dz = -dt \end{cases} \qquad 0 \leq t \leq 1.$$

Então:

- $\displaystyle\int_{\gamma_1} \vec{F}\, d\vec{r} = \int_0^1 P\,dx + Q\,dy + R\,dz$

$$= \int_0^1 e^t.\underbrace{\cos 0}_{=1}.dt + e^t.\underbrace{\operatorname{sen} 0}_{=0}.\,0 + 0.0 =$$

$$= \int_0^1 e^t\,dt = e^t\Big|_0^1 = e^1 - e^0 = \underline{\underline{e-1}}.$$

- $\displaystyle\int_{\gamma_2} \vec{F}\, d\vec{r} = \int_0^1 P\,dx + Q\,dy + R\,dz =$

$$= \int_0^1 e^x \cos y\,dx + e^x.\operatorname{sen} y\,dy + z\,dz =$$

$x = t$
$y = 0$
$z = 1-t$

$$= \int_0^1 e^t.\underbrace{\cos 0}_{=1}\,dt + e^t.\underbrace{\operatorname{sen} 0}_{=0}.\,0 + (1-t).(-dt) =$$

$$\int_0^1 e^t\,dt + \int_0^1 (t-1)\,dt$$

$$e^t\Big|_0^1 + \frac{(t-1)^2}{2}\Big|_0^1 = e^1 - e^0 + \frac{(1-1)^2}{2} - \frac{(0-1)^2}{2}$$

$$= e - 1 + 0 - \frac{1}{2} = e - \frac{3}{2}$$

Portanto,

$$\int_{\gamma} \vec{F}\, d\vec{r} = \int_{\gamma_1} + \int_{\gamma_2} = e - 1 + e - \frac{3}{2}$$

$$= 2e - \frac{5}{2}$$

$$\vec{F} = (P, Q, R)$$

(b) $$\operatorname{div} \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

$$= \frac{\partial}{\partial x}(e^x \cos y) + \frac{\partial}{\partial y}(e^x \operatorname{sen} y) + \frac{\partial}{\partial z}(z)$$

$$= e^x \cos y + e^x \cos y + 1 = 2 . e^x \cos y + 1 .$$

$$\operatorname{rot} \vec{F} = \left|\begin{matrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x \cos y & e^x \operatorname{sen} y & z \end{matrix}\right| \begin{matrix} \vec{i} & \vec{j} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ e^x \cos y & e^x . \operatorname{sen} y \end{matrix}$$

$$= 0\vec{i} + 0\vec{j} + e^x . \operatorname{sen} y . \vec{k} - (-e^x \operatorname{sen} y)\vec{k} - 0\vec{i} + 0\vec{j}$$

$$= 2 . e^x . \operatorname{sen} y . \vec{k} = (0, 0, 2e^x \operatorname{sen} y)$$