L5 CÁLCULO IV

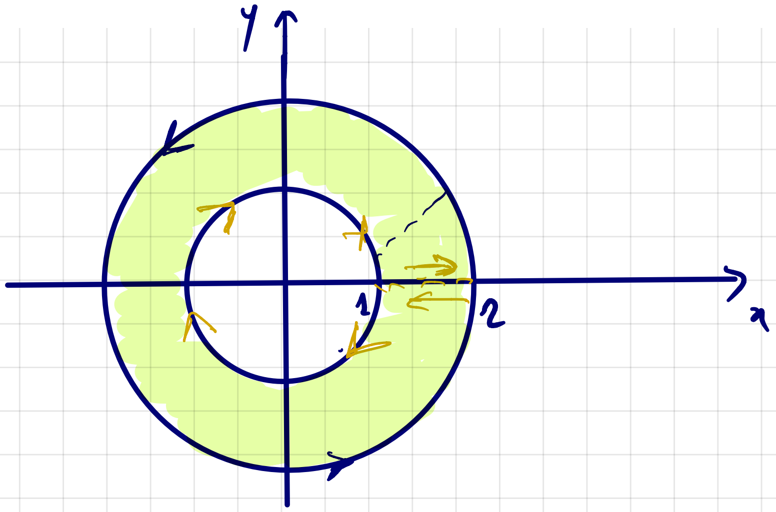
14/03/24 - AULA 23 (EXTRA)

8. Use o teorema de Green para calcular cada integral de linha a seguir, ao longo da curva dada com orientação positiva.

(a) $\int_\gamma e^y dx + 2xe^y dy$, onde γ é o quadrado de lados $x = 0$, $x = 1$, $y = 0$ e $y = 1$.

(b) $\int_\gamma (ye^{\sqrt{x}})dx + (2x + \cos y^2)dy$, onde γ é a fronteira da região delimitada pelas parábolas $y = x^2$ e $x = y^2$

(c) $\int_\gamma xe^{-2x}dx + (x^4 + 2x^2y^2)dy$, onde γ é a região entre as circunferências $x^2 + y^2 = 1$ e $x^2 + y^2 = 4$.



$$\oint_\gamma x e^{-2x} dx + (x^4 + 2x^2y^2)dy = \oint_\gamma P\,dx + Q\,dy$$

$$= \iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_\Omega (4x^3 + 4xy^2 - 0)\,dx\,dy$$

$$\int_{\theta=0}^{\theta=2\pi} \int_{\rho=1}^{\rho=2} (4.\rho^3 \cos^3\theta + 4\rho\cos\theta.\rho^2 \operatorname{sen}^2\theta)\,\rho\,d\rho\,d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{\rho=1}^{\rho=2} (4\rho^3 \cos^3\theta + 4\rho^3 \cos\theta . \operatorname{sen}^2\theta)\,\rho\,d\rho\,d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{\rho=1}^{\rho=2} 4\rho^3 \cos\theta \,\overbrace{(\cos^2\theta + \operatorname{sen}^2\theta)}^{=1}.\rho\,d\rho\,d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{\rho=1}^{\rho=2} 4\rho^3 \cos\theta \cdot \rho \, d\rho \, d\theta =$$

$$\int_{\theta=0}^{\theta=2\pi} \cos\theta \, d\theta \cdot \int_{\rho=1}^{\rho=2} 4\rho^4 \, d\rho = \underbrace{(\operatorname{sen}\theta)\Big|_{\theta=0}^{\theta=2\pi}} \cdot \frac{4\rho^5}{5}\Big|_{\rho=1}^{\rho=2} = 0$$

LISTA 06

9. Use o Teorema de Green na forma vetorial para provar a *primeira identidade de Green*:

$$\iint_\Omega f\Delta g dA = \oint_\gamma f(\nabla g) \cdot \vec{n} \, ds - \iint_\Omega \nabla f \cdot \nabla g dA,$$

onde Ω e γ satisfazem as hipóteses do Teorema de Green e as derivadas parciais apropriadas de f e g existem e são contínuas.

T. GREEN: NA FORMA VETORIAL (T. da divergência)

$$\oint_\gamma \vec{F} \cdot \vec{n} \, ds = \iint_\Omega \operatorname{div} \vec{F} \, dA.$$

Escreva $\vec{F} = f \cdot \nabla g$, onde $g, f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$

Assim,

$$\boxed{\oint_\gamma \vec{F} \cdot \vec{n} \, ds = \oint_\gamma f \cdot \nabla g \cdot \vec{n} \, ds \overset{\text{T. da divergência}}{=} \iint_\Omega \operatorname{div}(f \cdot \nabla g) \, dA,} \quad (*)$$

Note que

$$\underbrace{\operatorname{div}(f\cdot\nabla g)} = \operatorname{div}\left(f\cdot\left(\frac{\partial g}{\partial x},\frac{\partial g}{\partial y}\right)\right)$$

$$= \operatorname{div}\Big(\underbrace{f\cdot\frac{\partial g}{\partial x}}_{F_1},\ \underbrace{f\cdot\frac{\partial g}{\partial y}}_{F_2}\Big) =$$

$$= \frac{\partial}{\partial x}\left(f\cdot\frac{\partial g}{\partial x}\right) + \frac{\partial}{\partial y}\left(f\cdot\frac{\partial g}{\partial y}\right) =$$

$$= \underbrace{f\cdot\frac{\partial^2 g}{\partial x^2}} + \frac{\partial f}{\partial x}\cdot\frac{\partial g}{\partial x} + \underbrace{f\cdot\frac{\partial^2 g}{\partial y^2}} + \frac{\partial f}{\partial y}\cdot\frac{\partial g}{\partial y}$$

$$= f\cdot\underbrace{\left[\frac{\partial^2 g}{\partial x^2}+\frac{\partial^2 g}{\partial y^2}\right]}_{\Delta g} + \frac{\partial f}{\partial x}\cdot\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\cdot\frac{\partial g}{\partial y}$$

$$\underbrace{\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right)}_{\nabla f}\cdot\underbrace{\left(\frac{\partial g}{\partial x},\frac{\partial g}{\partial y}\right)}_{\nabla g}$$

$$= \underbrace{f\cdot\Delta g + \nabla f\cdot\nabla g}$$

Dessa forma, (*) fica,

$$\underbrace{\oint_\gamma f\cdot\nabla g\cdot\vec{n}\,ds} = \iint_\Omega \operatorname{div}(f\cdot\nabla g)\cdot dA =$$

$$\iint_\Omega (f\cdot\Delta g + \nabla f\cdot\nabla g)\cdot dA =$$

$$\iint_{\Omega} f \cdot \Delta g \, dA + \iint_{\Omega} \nabla f \cdot \nabla g \, dA$$

$$\Rightarrow \iint_{\Omega} f \cdot \Delta g \, dA = \oint_{\gamma} f \cdot \nabla g \cdot \vec{n} \, ds - \iint_{\Omega} \nabla f \cdot \nabla g \cdot dA .$$